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Ice B. Risteski*

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KEYWORDS: Quasicyclic complex vector functional equation

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A NEW CLASS OF QUASICYCLIC COMPLEX VECTOR FUNCTIONAL EQUATIONS

Ice B. RISTESKI

Abstract. For the first time in the literature a quasicyclic complex vector functional equation is introduced in the present paper. By a matrix method the general quasicyclic complex vector functional equation is solved, as well as its particular case for $n = 3$. This case is completely solved in an explicit form, and for every step of the solution examples are provided. Using a simple spectral property of compound matrices, a necessary and sufficient condition for stability of the quasicyclic complex vector functional equation considered is proved.

1. Introduction

The rapid development of the functional equations has found wide applications in the mathematical modelling in many sciences and engineering. A number of such examples are given in [1, 2]. This has led to considerable interest in the research of the additive and convex functions defined on linear spaces with semilinear topologies [4], stability of the functional equations in the sense of Ulam – Hyers – Rassias in some function spaces [5, 6], probability theory [8], stochastic models [9], actuarial mathematics [10], etc. as well as in many other branches of mathematics. Really, now it is very difficult to answer the question: where do functional equations not have applications?

The present paper is devoted to the study of a new class of quasicyclic complex vector functional equations. To the best of our knowledge, up to now this kind of complex vector functional equations has not been considered in the theory of functional equations, and we think that it will be of interest to study. For this reason we carried out our research with the goal to shed light on this insufficiently studied field of complex vector functional equations. The results presented here supplement and generalize some of our previous results [15].

2. Preliminaries

Let A be an $n \times n$ complex matrix. Suppose that by elementary transformations the matrix A is transformed into $A = P_1 D P_2$, where P_1 and P_2 are regular matrices and D is a diagonal matrix with diagonal entries 0 and 1 such that the number of the units is equal to the rank of the matrix A. The

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matrix $B = P_2^{-1}DP_1^{-1}$ satisfies the equality $ABA = A$. This means that the matrix equation $AXA = A$ has at least one solution for X.

If A satisfies the identity

$$A^r + k_1A^{r-1} + \cdots + k_{r-1}A = O,$$

where $k_{r-1} \neq 0$ and O is the zero $n \times n$ matrix, then the matrix

$$X = -\frac{1}{k_{r-1}}(A^{r-2} + k_1A^{r-3} + \cdots + k_{r-2}I),$$

where I is the unit $n \times n$ matrix, is also a solution of the equation $AXA = A$.

Now we recall the following theorem proved in [11].

Theorem 2.1. *If B satisfies the condition $ABA = A$, then*

1° $AX = O \iff X = (I-BA)Q$ *(X and Q are $n \times m$ matrices);*

2° $XA = O \iff X = Q(I-AB)$ *(X and Q are $m \times n$ matrices);*

3° $AXA = A \iff X = B + Q - BAQAB$ *(X and Q are $n \times n$ matrices);*

4° $AX = A \iff X = I + (I - BA)Q;$

5° $XA = A \iff X = I + Q(I - AB).$

Throughout this paper, $\mathcal{V}$ is an n-dimensional complex vector space. The vectors from $\mathcal{V}$ will be denoted by $\mathbf{Z}_i = (z_{i1}, \ldots, z_{in})^T$ $(1 \leq i \leq n)$, $\mathbf{U}$ and $\mathbf{V}$ are substitutional vectors in $\mathcal{V}$, and $\mathbf{O} = (0, 0, \ldots, 0)^T$ is the zero vector in $\mathcal{V}$.

Let $\otimes$ denote exterior product in $\mathcal{V}$ and let k $(1 \leq k \leq n)$ be an integer. With respect to the canonical basis in the k-th exterior product space $\bigotimes^k \mathcal{V}$, the k-th additive compound matrix $A^{[k]}$ of A is a linear operator on $\bigotimes^k \mathcal{V}$ whose definition on a decomposable element $\mathbf{Z}_1 \otimes \cdots \otimes \mathbf{Z}_k$ is

$$A^{[k]}(\mathbf{Z}_1 \otimes \cdots \otimes \mathbf{Z}_k) = \sum_{i=1}^{k} \mathbf{Z}_1 \otimes \cdots \otimes A\mathbf{Z}_i \otimes \cdots \otimes \mathbf{Z}_k. \tag{2.1}$$

For any integer $i = 1, 2, \ldots, \binom{n}{k}$, let $(i) = (i_1, \ldots, i_k)$ be the i-th member in the lexicographic ordering of integer k-tuples such that $1 \leq i_1 < \cdots < i_k \leq n$. Then the (i, j)-th entry of the matrix $A^{[k]} = [q_{ij}]$ is

(2.2)

$$q_{ij} = \begin{cases} a_{i_1i_1} + \cdots + a_{i_ki_k} & \text{if } (i) = (j), \\ (-1)^{m+s}a_{j_mi_s} & \text{if exactly one entry } i_s \text{ of } (i) \text{ does not occur in } (j) \text{ and } j_m \text{ does not occur in } (i), \\ 0 & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases}$$

As special cases we have $A^{[1]} = A$ and $A^{[n]} = \operatorname{tr} A$ [7].

Let $\sigma(A) = \{\lambda_i,\ 1 \leq i \leq n\}$ be the spectrum of A. Then the spectrum of $A^{[k]}$ is $\sigma(A^{[k]}) = \{\lambda_{i_1} + \cdots + \lambda_{i_k},\ 1 \leq i_1 < \cdots < i_k \leq n\}$.

Let $|\cdot|$ denote a vector norm in $\mathcal{V}^n$. The Lozinskiĭ measure μ on $\mathcal{V}^n$ with respect to $|\cdot|$ is defined by

$$\mu(A) = \lim_{\rho \to 0^+} \frac{|I + \rho A| - 1}{\rho}. \tag{2.3}$$

The Lozinskiĭ measures of $A = [a_{ij}]_{n\times n}$ with respect to the three common norms

$$\begin{aligned} |\mathbf{Z}|_\infty &= \sup_i |z_i|, \\ |\mathbf{Z}|_1 &= \sum_i |z_i|, \\ |\mathbf{Z}|_2 &= \left(\sum_i |z_i|^2 \right)^{1/2} \end{aligned}$$

are

$$\begin{aligned} \mu_\infty(A) &= \sup_i \left(\operatorname{Re} a_{ii} + \sum_{k,\, k\neq i} |a_{ik}| \right), \\ \mu_1(A) &= \sup_k \left(\operatorname{Re} a_{kk} + \sum_{i,\, i\neq k} |a_{ik}| \right), \\ \mu_2(A) &= \operatorname{stab} \left(\frac{A + A^*}{2} \right), \end{aligned} \tag{2.4}$$

where

$$\operatorname{stab}(A) = \max\{\operatorname{Re}\lambda,\ \lambda \in \sigma(A)\}$$

is the stability modulus of the matrix A, and A^* denotes the Hermitian adjoint of A [3, p. 41].

3. Solution of a quasicyclic functional equation

Here we will prove the following result.

Theorem 3.1. *The quasicyclic complex vector functional equation*

$$\begin{aligned} E(f) &\equiv \sum_{i=1}^{n} a_i f(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-1}) \\ &= \sum_{i=1}^{n} \alpha_i f(\mathbf{Z}_i, \mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-2}) \quad (n > 1) \end{aligned} \tag{3.1}$$

$$(\mathbf{Z}_{n+i} \equiv \mathbf{Z}_i)$$

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where a_i, α_i $(1 \le i \le n)$ are complex constants, has a solution if the right-hand side of (3.1) *satisfies*

$$[AC+I]\Lambda \begin{bmatrix} g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{n-1}) \\ g(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_n) \\ \vdots \\ g(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2}) \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \end{bmatrix}, \tag{3.2}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ a_n & a_1 & \ldots & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & \ldots & a_1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_n & \alpha_1 & \ldots & \alpha_{n-1} \\ \vdots & & & \\ \alpha_2 & \alpha_3 & \ldots & \alpha_1 \end{bmatrix},$$

$g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{n-1}) = f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{n-1})$, *$C$ is any nonzero $n\times n$ cyclic matrix with complex constant entries satisfying $ACA+A=O$, O is the $n\times n$ zero matrix and I is the $n\times n$ unit matrix.*

If the equality (3.2) *holds for some C, then the general solution of the equation* (3.1) *is given by the following formula*

$$\begin{aligned} \begin{bmatrix} f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) \\ f(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_n,\mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \end{bmatrix} &= B \begin{bmatrix} h(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) \\ h(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_n,\mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \end{bmatrix} \\ &\quad - C\Lambda \begin{bmatrix} g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_{n-1}) \\ g(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_n) \\ \vdots \\ g(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2}) \end{bmatrix}, \end{aligned} \tag{3.3}$$

where the nonzero $n\times n$ cyclic matrix B given by

$$B = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \\ b_n & b_1 & \ldots & b_{n-1} \\ \vdots & & & \\ b_2 & b_3 & \ldots & b_1 \end{bmatrix}$$

satisfies the condition

$$AB = O$$

and h is an arbitrary complex vector function $\mathcal{V}^n \mapsto \mathcal{V}$.

Proof. By a cyclic permutation of the vectors in (3.1) we get

$$\begin{aligned}
&a_1 f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) + a_2 f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_1) + \cdots \\
&\quad + a_n f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \\
&= \alpha_1 g(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}) + \alpha_2 g(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n) + \cdots \\
&\quad + \alpha_n g(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-2}), \\
&a_n f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) + a_1 f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_1) + \cdots \\
&\quad + a_{n-1} f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \\
&= \alpha_n g(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}) + \alpha_1 g(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n) + \cdots \\
&\quad + \alpha_{n-1} g(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-2}), \\
&\quad \vdots \\
&a_2 f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) + a_3 f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_1) + \cdots \\
&\quad + a_1 f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \\
&= \alpha_2 g(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}) + \alpha_3 g(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n) + \cdots \\
&\quad + \alpha_1 g(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-2}),
\end{aligned}$$

i.e., in a matrix form

$$AF = \Lambda G, \tag{3.4}$$

where

$$F = \begin{bmatrix} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} g(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{n-1}) \\ g(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n) \\ \vdots \\ g(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-2}) \end{bmatrix}.$$

We suppose that equation (3.4) has a solution F and C satisfies $ACA + A = O$. Then

$$[AC + I]\Lambda G = [AC + I]AF = [ACA + A]F = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \end{bmatrix},$$

i.e., equation (3.2) must be satisfied. Conversely, let equation (3.2) hold for some cyclic matrix C. Then $-C\Lambda G$ is easily seen to be a solution of equation (3.4):

$$A[-C\Lambda G] = -[AC + I]\Lambda G + I\Lambda G = I\Lambda G = \Lambda G.$$

Now let us prove that equality (3.3) gives the general solution of the equation (3.1).

Let f be a solution of the equation (3.1), which we will write in the form

$$E(f) = L(g). \tag{3.5}$$

We denote by f_h the general solution of the equation $E(f) = \mathbf{O}$, and by f_p we denote a particular solution of the equation (3.5).

Then $f = f_h + f_p$ is the general solution of the equation (3.5). Indeed,

$$E(f_h + f_p) = E(f_h) + E(f_p) = L(g).$$

On the other hand, let f be an arbitrary solution of the equation (3.5). Then

$$E(f - f_p) = E(f) - E(f_p) = L(g) - L(g) = \mathbf{O},$$

i.e., $f - f_p$ is a solution of the associated homogeneous equation. So there exists a specialization $\bar{f}_h$ of the expression f_h such that

$$f - f_p = \bar{f}_h, \quad i.e., \quad f = \bar{f}_h + f_p.$$

Thus $f_h + f_p$ includes all solutions of the equation (3.5).

The general solution of the homogeneous equation $E(f) = \mathbf{O}$ given in a matrix form is BH, where

$$H = \begin{bmatrix} h(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ h(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix},$$

and a particular solution of the equation $E(f) = L(g)$ in a matrix form is $-C\Lambda G$, then $F = BH - C\Lambda G$ includes all solutions of the nonhomogeneous equation.

On the other hand, every function of the form (3.3) satisfies the functional equation (3.1). $\square$

4. Analysis of a particular case

In this section we will develop Theorem 3.1 in whole for the particular case $n = 3$, so that for every step of the solution we will give examples for a better description of the proof. For this case the quasicyclic complex vector functional equation (3.1) takes the form

$$\begin{aligned} &a_1 f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + a_2 f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + a_3 f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \ &\alpha_1 f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + \alpha_2 f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + \alpha_3 f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \quad (f: \ \mathcal{V}^3 \mapsto \mathcal{V}), \end{aligned} \tag{4.1}$$

where $a_i, \ \alpha_i$ $(1 \le i \le 3)$ are complex constants.

For the equation (4.1) we suppose that $|\alpha_1| + |\alpha_2| + |\alpha_3| > 0$ and $|a_1| + |a_2| + |a_3| > 0$. In the case $\alpha_1 = \alpha_2 = \alpha_3 = 0$ the quasicyclic functional equation (4.1) transforms into cyclic functional equations whose solution is given in [11].

In the solution of the above equation we will use the techniques given in [11, 12, 13, 14, 16]. If we permute cyclically the vectors in equation (4.1), we get

$$\begin{aligned}
&a_1 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)+a_3 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\
&= \alpha_1 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_2 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_3 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
&a_1 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)+a_2 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_3 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)\\
&= \alpha_1 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_2 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_3 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3).
\end{aligned}\tag{4.3}$$

The determinant for the system of equations (4.1), (4.2) and (4.3) is

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3\\ a_3 & a_1 & a_2\\ a_2 & a_3 & a_1\end{vmatrix}.$$

We shall note the identity

$$\Delta = \frac{1}{2}(a_1+a_2+a_3)\big[(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2\big].\tag{4.4}$$

We may distinguish the following two cases:

1°. Let $\Delta \neq 0$. From (4.1), (4.2) and (4.3) we obtain

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\begin{vmatrix} \alpha_1 F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2 F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_3 F(\mathbf{Z}_3,\mathbf{Z}_1) & a_2 & a_3\\ \alpha_1 F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_2 F(\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_3 F(\mathbf{Z}_1,\mathbf{Z}_2) & a_1 & a_2\\ \alpha_1 F(\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_2 F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_3 F(\mathbf{Z}_2,\mathbf{Z}_3) & a_3 & a_1\end{vmatrix},$$

where $F:\ \mathcal{V}^2 \mapsto \mathcal{V}$ is defined by $F(\mathbf{U},\mathbf{V})=\frac{1}{\Delta}f(\mathbf{U},\mathbf{U},\mathbf{V})$.

If we introduce the notation

$$\Delta_1=\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3\\ a_3 & a_1 & a_2\\ a_2 & a_3 & a_1\end{vmatrix},\quad \Delta_2=\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3\\ a_2 & a_3 & a_1\\ a_1 & a_2 & a_3\end{vmatrix},\quad \Delta_3=\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3\\ a_1 & a_2 & a_3\\ a_3 & a_1 & a_2\end{vmatrix},$$

then we can write

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\Delta_1 F(\mathbf{Z}_1,\mathbf{Z}_2)+\Delta_2 F(\mathbf{Z}_2,\mathbf{Z}_3)+\Delta_3 F(\mathbf{Z}_3,\mathbf{Z}_1).\tag{4.5}$$

For (4.5) to be a solution of the equation (4.1), the following condition must be satisfied:

$$\begin{aligned}
&\alpha_1\big[(\Delta-\Delta_2)F(\mathbf{Z}_1,\mathbf{Z}_2)-\Delta_3 F(\mathbf{Z}_2,\mathbf{Z}_1)-\Delta_1 F(\mathbf{Z}_1,\mathbf{Z}_1)\big]\\
+\ &\alpha_2\big[(\Delta-\Delta_2)F(\mathbf{Z}_2,\mathbf{Z}_3)-\Delta_3 F(\mathbf{Z}_3,\mathbf{Z}_2)-\Delta_1 F(\mathbf{Z}_2,\mathbf{Z}_2)\big]\\
+\ &\alpha_3\big[(\Delta-\Delta_2)F(\mathbf{Z}_3,\mathbf{Z}_1)-\Delta_3 F(\mathbf{Z}_1,\mathbf{Z}_3)-\Delta_1 F(\mathbf{Z}_3,\mathbf{Z}_3)\big]=\mathbf{O}.
\end{aligned}\tag{4.6}$$

By a cyclic permutation of the variables $\mathbf{Z}_1,\ \mathbf{Z}_2,\ \mathbf{Z}_3$ in (4.6) we obtain two new equations. The system of these three equations has a nontrivial solution with respect to $(\Delta-\Delta_2)F(\mathbf{Z}_i,\mathbf{Z}_{i+1})-\Delta_3F(\mathbf{Z}_{i+1},\mathbf{Z}_i)-\Delta_1F(\mathbf{Z}_i,\mathbf{Z}_i)$, $i=1,2,3$ (with the convention $\mathbf{Z}_4\equiv\mathbf{Z}_1$) if the following condition is satisfied:

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \alpha_2 \end{vmatrix}=0. \tag{4.7}$$

By virtue of an equality of the type of (4.4) this is true if

$$\alpha_1+\alpha_2+\alpha_3=0 \quad \text{or} \quad (\alpha_1-\alpha_2)^2+(\alpha_2-\alpha_3)^2+(\alpha_3-\alpha_1)^2=0.$$

First we will consider the case

$$\alpha_1+\alpha_2+\alpha_3=0. \tag{4.8}$$

Since $a_1+a_2+a_3\neq 0$ (because of the assumption $\Delta\neq 0$ and (4.4)), by putting into the equation (4.1) $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{Z}_3$ we derive $F(\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}$. By using the last equality, the equation (4.6) for $\mathbf{Z}_3=\mathbf{Z}_1$ becomes

$$[\alpha_1(\Delta-\Delta_2)-\alpha_2\Delta_3]F(\mathbf{Z}_1,\mathbf{Z}_2)-[\alpha_1\Delta_3-\alpha_2(\Delta-\Delta_2)]F(\mathbf{Z}_2,\mathbf{Z}_1)=\mathbf{O}. \tag{4.9}$$

If we change the places of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, the equation (4.9) is transformed into

$$-[\alpha_1\Delta_3-\alpha_2(\Delta-\Delta_2)]F(\mathbf{Z}_1,\mathbf{Z}_2)+[\alpha_1(\Delta-\Delta_2)-\alpha_2\Delta_3]F(\mathbf{Z}_2,\mathbf{Z}_1)=\mathbf{O}. \tag{4.10}$$

Let

$$(\alpha_1^2-\alpha_2^2)[(\Delta-\Delta_2)^2-\Delta_3^2]\neq 0,$$

then from (4.9) and (4.10) it follows that $F(\mathbf{Z}_1,\mathbf{Z}_2)=\mathbf{O}$ and then from (4.5) $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\mathbf{O}$.

The condition

$$(\alpha_1^2-\alpha_2^2)[(\Delta-\Delta_2)^2-\Delta_3^2]=0. \tag{4.11}$$

implies $(\Delta-\Delta_2)^2-\Delta_3^2=0$. In fact, suppose that the last equality is not true. Now, if we set $\mathbf{Z}_3=\mathbf{Z}_2$ into (4.6), we obtain

$$(\alpha_1^2-\alpha_3^2)[(\Delta-\Delta_2)^2-\Delta_3^2]=0.$$

The last equality, with (4.11), gives $\alpha_1^2=\alpha_2^2=\alpha_3^2$ which, by virtue of the assumption (4.8), yields $\alpha_1=\alpha_2=\alpha_3=0$, which contradicts the hypothesis $|\alpha_1|+|\alpha_2|+|\alpha_3|>0$.

Thus, we have $\Delta-\Delta_2=\pm\Delta_3$. For the case $\Delta-\Delta_2=\Delta_3$ $(\neq 0)$, the equation (4.9) yields

$$(\alpha_1-\alpha_2)\Delta_3[F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_1)]=\mathbf{O},$$

so that, for $\alpha_1\neq\alpha_2$, we have

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1), \tag{4.12}$$

where G is an arbitrary function $\mathcal{V}^2\mapsto\mathcal{V}$ such that $G(\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$.

If $\alpha_1 = \alpha_2$, then necessarily we must have $\alpha_1 \neq \alpha_3$ (because otherwise we would have $\alpha_1 = \alpha_2 = \alpha_3 = 0$) and by a procedure analogous to the above one we obtain (4.12).

Example 4.1. The general solution of the functional equation

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &- 2f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ &= -5f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + 4f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = G(\mathbf{Z}_1, \mathbf{Z}_2) &+ G(\mathbf{Z}_2, \mathbf{Z}_1) - 2G(\mathbf{Z}_2, \mathbf{Z}_3) \\ &- 2G(\mathbf{Z}_3, \mathbf{Z}_2) + G(\mathbf{Z}_3, \mathbf{Z}_1) + G(\mathbf{Z}_1, \mathbf{Z}_3), \end{aligned}$$

where G is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$ such that $G(\mathbf{Z}_1, \mathbf{Z}_1) \equiv \mathbf{O}$.

Let $\Delta - \Delta_2 = -\Delta_3$ ($\neq 0$), from (4.9) we obtain

$$(\alpha_1 + \alpha_2)\Delta_3[F(\mathbf{Z}_1, \mathbf{Z}_2) + F(\mathbf{Z}_2, \mathbf{Z}_1)] = \mathbf{O}. \tag{4.13}$$

For $\alpha_1 + \alpha_2 \neq 0$, the general solution of the equation (4.13) is given by

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = G(\mathbf{Z}_1, \mathbf{Z}_2) - G(\mathbf{Z}_2, \mathbf{Z}_1), \qquad G : \mathcal{V}^2 \mapsto \mathcal{V}. \tag{4.14}$$

If $\alpha_1 + \alpha_2 = 0$, then from (4.8) we deduce $\alpha_3 = 0$, and then $\alpha_1 + \alpha_3 = \alpha_1 \neq 0$ and we obtain (4.14) by an analogous procedure.

Example 4.2. The functional equation

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + 2f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) &+ 2f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) \\ &- 3f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

has the function

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 5G(\mathbf{Z}_1, \mathbf{Z}_2) &- 5G(\mathbf{Z}_2, \mathbf{Z}_1) - 4G(\mathbf{Z}_2, \mathbf{Z}_3) \\ &+ 4G(\mathbf{Z}_3, \mathbf{Z}_2) - G(\mathbf{Z}_3, \mathbf{Z}_1) + G(\mathbf{Z}_1, \mathbf{Z}_3), \end{aligned}$$

$G : \mathcal{V}^2 \mapsto \mathcal{V}$, as a general solution.

The condition (4.6), for the case $\Delta_3 = \Delta - \Delta_2 = 0$, is satisfied for every function $F(\mathbf{Z}_1, \mathbf{Z}_2)$ with the property $F(\mathbf{Z}_1, \mathbf{Z}_1) \equiv \mathbf{O}$.

Example 4.3. The general solution of the functional equation

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = f&(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) \\ - f&(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

is given by

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = F(\mathbf{Z}_1, \mathbf{Z}_2) - F(\mathbf{Z}_2, \mathbf{Z}_3),$$

where F is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$ such that $F(\mathbf{Z}_1, \mathbf{Z}_1) \equiv \mathbf{O}$.

If $(\Delta - \Delta_2)^2 \neq \Delta_3^2$, then, as it was mentioned above, the equations (4.9) and (4.10) have the trivial solution as the general solution. According to (4.5) we obtain

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \equiv \mathbf{O}.$$

Example 4.4. The functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) = f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1)$$

has $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \equiv \mathbf{O}$ as the unique solution.

Now we suppose that (4.7) is satisfied but (4.8) is not. This means that

$$(\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_1)^2 = 0. \tag{4.15}$$

Since $\alpha_1,\ \alpha_2,\ \alpha_3$ are complex constants, (4.15) does not necessarily imply $\alpha_1 = \alpha_2 = \alpha_3$. If, however, the constants $\alpha_1,\ \alpha_2,\ \alpha_3$ satisfy the equality (4.15) and two of them are equal, then the third one is also equal to the other two. Now we will consider the case

$$\alpha_1 = \alpha_2 = \alpha_3 \neq 0.$$

It immediately follows that

$$\Delta_1 = \Delta_2 = \Delta_3\ (\neq 0).$$

The quasicyclic equation (4.6) implies

$$(\Delta - \Delta_1)F(\mathbf{Z}_1, \mathbf{Z}_2) - \Delta_1 F(\mathbf{Z}_2, \mathbf{Z}_1) - \Delta_1 F(\mathbf{Z}_1, \mathbf{Z}_1) = \Delta_1 P(\mathbf{Z}_1) - \Delta_1 P(\mathbf{Z}_2), \tag{4.16}$$

where P is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

For $\alpha_1 = \alpha_2 = \alpha_3$ and $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$ the equation (4.1) becomes

$$(a_1 + a_2 + a_3 - 3\alpha_1)F(\mathbf{Z}_1, \mathbf{Z}_1) = \mathbf{O}.$$

Let $a_1 + a_2 + a_3 = 3\alpha_1$, then $3\Delta_1 = \Delta$ and the equality (4.16) takes the form

$$2F(\mathbf{Z}_1, \mathbf{Z}_2) - F(\mathbf{Z}_2, \mathbf{Z}_1) = P(\mathbf{Z}_1) - P(\mathbf{Z}_2) + R(\mathbf{Z}_1),$$

where $R(\mathbf{Z}_1) = F(\mathbf{Z}_1, \mathbf{Z}_1)$.

By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ it follows that

$$-F(\mathbf{Z}_1, \mathbf{Z}_2) + 2F(\mathbf{Z}_2, \mathbf{Z}_1) = P(\mathbf{Z}_2) - P(\mathbf{Z}_1) + R(\mathbf{Z}_2).$$

From the last two equalities we obtain

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = \frac{1}{3}[P(\mathbf{Z}_1) - P(\mathbf{Z}_2) + 2R(\mathbf{Z}_1) + R(\mathbf{Z}_2)].$$

By using the last equality, from (4.5) it follows that

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = Q(\mathbf{Z}_1) + Q(\mathbf{Z}_2) + Q(\mathbf{Z}_3), \tag{4.17}$$

where $Q(\mathbf{Z}_1) = \Delta_1 R(\mathbf{Z}_1)$.

Example 4.5. The general solution of the equation

$$2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) = f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)$$

is given by the expression (4.17), where Q is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

Let $a_1 + a_2 + a_3 \neq 3\alpha_1$, then $F(\mathbf{Z}_1,\mathbf{Z}_1) \equiv \mathbf{O}$ and the formula (4.16) yields

$$(\Delta - \Delta_1)F(\mathbf{Z}_1,\mathbf{Z}_2) - \Delta_1 F(\mathbf{Z}_2,\mathbf{Z}_1) = \Delta_1 P(\mathbf{Z}_1) - \Delta_1 P(\mathbf{Z}_2). \tag{4.18}$$

From the last equality by a permutation of the variables we obtain

$$-\Delta_1 F(\mathbf{Z}_1,\mathbf{Z}_2) + (\Delta - \Delta_1)F(\mathbf{Z}_2,\mathbf{Z}_1) = \Delta_1 P(\mathbf{Z}_2) - \Delta_1 P(\mathbf{Z}_1).$$

The determinant of the system consisting of the last two equations is $\Delta(\Delta - 2\Delta_1)$. If $\Delta \neq 2\Delta_1$, the solution of the last two equations is

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = \frac{\Delta_1}{\Delta}[P(\mathbf{Z}_1) - P(\mathbf{Z}_2)].$$

Then the equality (4.5) gives

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}.$$

Example 4.6. The functional equation

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + 2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) + f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

has the general solution $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}$.

Let $\Delta = 2\Delta_1$, then from (4.18) we obtain

$$F(\mathbf{Z}_1,\mathbf{Z}_2) - P(\mathbf{Z}_1) = F(\mathbf{Z}_2,\mathbf{Z}_1) - P(\mathbf{Z}_2).$$

The general solution for the last equation is

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = P(\mathbf{Z}_1) + G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1), \tag{4.19}$$

where $P:\ \mathcal{V} \mapsto \mathcal{V}$ and $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ are arbitrary functions such that

$$G(\mathbf{Z}_1,\mathbf{Z}_1) = -\frac{1}{2}P(\mathbf{Z}_1).$$

According to the last relation, the equality (4.19) takes the form

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1) - 2G(\mathbf{Z}_1,\mathbf{Z}_1)$$

and the formula (4.5) becomes

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1) - 2G(\mathbf{Z}_1,\mathbf{Z}_1) \\ &+ G(\mathbf{Z}_2,\mathbf{Z}_3) + G(\mathbf{Z}_3,\mathbf{Z}_2) - 2G(\mathbf{Z}_2,\mathbf{Z}_2) \\ &+ G(\mathbf{Z}_3,\mathbf{Z}_1) + G(\mathbf{Z}_1,\mathbf{Z}_3) - 2G(\mathbf{Z}_3,\mathbf{Z}_3), \end{aligned} \tag{4.20}$$

where we have replaced $G(\mathbf{Z}_1,\mathbf{Z}_2)\Delta_1$ by $G(\mathbf{Z}_1,\mathbf{Z}_2)$.

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Example 4.7. All solutions of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + 2f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + 3f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \ & 3f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + 3f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + 3f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

have the form (4.20), where $G: \ \mathcal{V}^2 \mapsto \mathcal{V}$ is an arbitrary function.

Now we suppose that the equality (4.15) is satisfied but the constants $\alpha_1, \ \alpha_2, \ \alpha_3$ are all distinct. This means that at least one of them is not real. In this case, each one of the three constants can be expressed in terms of the other two as follows, say,

$$\alpha_3 = [\alpha_1 + \alpha_2 \pm i(\alpha_1 - \alpha_2)\sqrt{3}]/2.$$

If we denote by ω a primitive third root of 1 ($\frac{-1 \pm i\sqrt{3}}{2}$), the last equality can be written as

$$\alpha_1 \omega + \alpha_2 \omega^2 + \alpha_3 = 0, \quad \alpha_1 \neq \alpha_2. \tag{4.21}$$

In the sequel we will often use the equalities

$$\omega^3 = 1 \qquad \text{and} \qquad \omega^2 + \omega + 1 = 0.$$

So we suppose that (4.21) holds. If we put into equation (4.1) $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$, we obtain

$$(a_1 + a_2 + a_3 - \alpha_1 - \alpha_2 - \alpha_3)F(\mathbf{Z}_1, \mathbf{Z}_1) = \mathbf{O}.$$

First we shall consider the case

$$a_1 + a_2 + a_3 \neq \alpha_1 + \alpha_2 + \alpha_3.$$

In this case $F(\mathbf{Z}_1, \mathbf{Z}_1) = \mathbf{O}$ and, as above, from equation (4.6) we deduce (4.9). The further investigation of this case proceeds completely as above, so we shall just recall the results.

If $(\Delta - \Delta_2)^2 \neq \Delta_3^2$, then $F(\mathbf{Z}_1, \mathbf{Z}_2) \equiv \mathbf{O}$ and $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \equiv \mathbf{O}$.

If $\Delta - \Delta_2 = \Delta_3 \neq 0$, then $F(\mathbf{Z}_1, \mathbf{Z}_2)$ is given by the equality (4.12), where $G(\mathbf{Z}_1, \mathbf{Z}_1) \equiv \mathbf{O}$.

If $\Delta - \Delta_2 = -\Delta_3 \neq 0$, then $F(\mathbf{Z}_1, \mathbf{Z}_2)$ is given by the equality (4.14).

If $\Delta - \Delta_2 = \Delta_3 = 0$, then equation (4.6) is satisfied by any function $F(\mathbf{Z}_1, \mathbf{Z}_2)$ with the property $F(\mathbf{Z}_1, \mathbf{Z}_1) \equiv \mathbf{O}$.

Then in all these cases the general solution $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ of the equation (4.1) can be expressed in terms of $F(\mathbf{Z}_1, \mathbf{Z}_2)$ using the equality (4.5).

Example 4.8. The functional equation

$$\begin{aligned} & f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ & \qquad = i\sqrt{3}f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - i\sqrt{3}f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + 3f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

has $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = \mathbf{O}$ as the unique solution.

Example 4.9. The general solution of the functional equation

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)=-\omega^2 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)$$

is given by the formula

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \omega[G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1)]-\omega[G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2)] \\ &\quad +(2+\omega)[G(\mathbf{Z}_3,\mathbf{Z}_1)+G(\mathbf{Z}_1,\mathbf{Z}_3)], \end{aligned}$$

where $G:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function satisfying $G(\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$.

Example 4.10. The general solution of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+(4-i\sqrt{3})f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = \ & i\sqrt{3}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-i\sqrt{3}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+3f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=(6-2i\sqrt{3})[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)] \\ + \ & (3+5i\sqrt{3})[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)]+6[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)], \end{aligned}$$

where $G:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function.

Example 4.11. The general solution of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) \\ = \ & -\omega f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+(1-\omega)f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\omega F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_3),$$

where $F:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function.

Now let us consider the case

$$a_1+a_2+a_3=\alpha_1+\alpha_2+\alpha_3\neq 0. \tag{4.22}$$

Then

$$\begin{aligned} \Delta_1 &= (a_1+a_2+a_3)(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3-a_1a_2-a_1a_3-a_2a_3), \\ \Delta_2 &= (a_1+a_2+a_3)(a_1\alpha_2+a_2\alpha_3+a_3\alpha_1-a_1a_2-a_1a_3-a_2a_3), \\ \Delta_3 &= (a_1+a_2+a_3)(a_1\alpha_3+a_2\alpha_1+a_3\alpha_2-a_1a_2-a_1a_3-a_2a_3). \end{aligned}$$

We may notice the relation

$$\Delta=\Delta_1+\Delta_2+\Delta_3. \tag{4.23}$$

From the quasicyclic equation (4.6) by virtue of the relations (4.21) and (4.22) we obtain

$$(\Delta-\Delta_2)F(\mathbf{Z}_1,\mathbf{Z}_2)-\Delta_3F(\mathbf{Z}_2,\mathbf{Z}_1)-\Delta_1F(\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}.$$

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In view of (4.23) this equation can be written as

$$(\Delta_1 + \Delta_3)F(\mathbf{Z}_1, \mathbf{Z}_2) - \Delta_3 F(\mathbf{Z}_2, \mathbf{Z}_1) = \Delta_1 R(\mathbf{Z}_1),$$

where $R(\mathbf{Z}_1) = F(\mathbf{Z}_1, \mathbf{Z}_1)$. By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we obtain

$$-\Delta_3 F(\mathbf{Z}_1, \mathbf{Z}_2) + (\Delta_1 + \Delta_3)F(\mathbf{Z}_2, \mathbf{Z}_1) = \Delta_1 R(\mathbf{Z}_2).$$

The determinant of this system is $(\Delta_1 + 2\Delta_3)\Delta_1$. If it is not zero, then

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = \frac{(\Delta_1 + \Delta_3)R(\mathbf{Z}_1) + \Delta_3 R(\mathbf{Z}_2)}{\Delta_1 + 2\Delta_3}.$$

From (4.5) it follows that

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = {} & (\Delta_1^2 + \Delta_1\Delta_3 + \Delta_3^2)Q(\mathbf{Z}_1) \\ & + (\Delta_1\Delta_2 + \Delta_1\Delta_3 + \Delta_2\Delta_3)Q(\mathbf{Z}_2) + \Delta_3\Delta Q(\mathbf{Z}_3), \end{aligned}$$

where

$$Q(\mathbf{Z}_1) = \frac{R(\mathbf{Z}_1)}{\Delta_1 + 2\Delta_3}.$$

Example 4.12. The general solution of the equation

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + 2f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = {} & i\sqrt{3}f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - i\sqrt{3}f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) \\ & + 3f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

is given by the expression

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = (4 - i\sqrt{3})Q(\mathbf{Z}_1) + Q(\mathbf{Z}_2) + (1 - 2i\sqrt{3})Q(\mathbf{Z}_3),$$

where Q is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

Let $\Delta_1 = 0,\ \Delta_3 \neq 0$. Then

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = F(\mathbf{Z}_2, \mathbf{Z}_1),$$

thus

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = G(\mathbf{Z}_1, \mathbf{Z}_2) + G(\mathbf{Z}_2, \mathbf{Z}_1),$$

where G is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$, and $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ is given by formula (4.5).

Example 4.13. The general solution of the functional equation

$$\begin{aligned} & (\omega - 1)[f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1)] \\ = \; & -f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + \omega f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + (\omega - 1)f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = \omega[G(\mathbf{Z}_2, \mathbf{Z}_3) + G(\mathbf{Z}_3, \mathbf{Z}_2)] - G(\mathbf{Z}_1, \mathbf{Z}_3) - G(\mathbf{Z}_3, \mathbf{Z}_1),$$

where $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ is an arbitrary function.

Now let us consider the case

$$\Delta_1 = \Delta_3 = 0, \quad \Delta = \Delta_2 \neq 0.$$

Now we shall use

Lemma 4.1. *Let* $\Delta \neq 0$. *Then the system*

$$\Delta_1 = 0, \quad \Delta_2 - \Delta = 0, \quad \Delta_3 = 0 \tag{4.24}$$

implies $\alpha_1 = a_3,\ \alpha_2 = a_1,\ \alpha_3 = a_2$.

Proof. The system (4.24) can be written in the form

$$\begin{aligned} A_{11}(\alpha_1 - a_3) + A_{12}(\alpha_2 - a_1) + A_{13}(\alpha_3 - a_2) = 0, \\ A_{21}(\alpha_1 - a_3) + A_{22}(\alpha_2 - a_1) + A_{23}(\alpha_3 - a_2) = 0, \\ A_{31}(\alpha_1 - a_3) + A_{32}(\alpha_2 - a_1) + A_{33}(\alpha_3 - a_2) = 0, \end{aligned} \tag{4.25}$$

where A_{ij} is the cofactor of the element a_{ij} $(1 \leq i,\, j \leq 3)$ of the determinant Δ. The system (4.25) is a homogeneous linear system with respect to $\alpha_1 - a_3,\ \alpha_2 - a_1,\ \alpha_3 - a_2$. Its determinant is $\Delta^2 \neq 0$, so it has only the zero solution. □

Thus we obtain $\alpha_1 = a_3,\ \alpha_2 = a_1,\ \alpha_3 = a_2$. So

$$0 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \alpha_2 \end{vmatrix} = -\,\Delta \neq 0$$

which is a contradiction.

Now we suppose that $\Delta_1 = -\,2\Delta_3 \neq 0$. Then

$$F(\mathbf{Z}_1, \mathbf{Z}_2) + F(\mathbf{Z}_2, \mathbf{Z}_1) = 2R(\mathbf{Z}_1). \tag{4.26}$$

By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we obtain

$$F(\mathbf{Z}_2, \mathbf{Z}_1) + F(\mathbf{Z}_1, \mathbf{Z}_2) = 2R(\mathbf{Z}_2). \tag{4.27}$$

From (4.26) and (4.27) we get

$$R(\mathbf{Z}_1) = R(\mathbf{Z}_2) = \mathcal{C}.$$

Thus (4.26) takes on the form

$$[F(\mathbf{Z}_1, \mathbf{Z}_2) - \mathcal{C}] + [F(\mathbf{Z}_2, \mathbf{Z}_1) - \mathcal{C}] = \mathbf{O}$$

which implies that

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = G(\mathbf{Z}_1, \mathbf{Z}_2) - G(\mathbf{Z}_2, \mathbf{Z}_1) + \mathcal{C},$$

where $G :\ \mathcal{V}^2 \mapsto \mathcal{V}$ is an arbitrary function and $\mathcal{C}$ is an arbitrary constant vector.

Now from (4.5) we find

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = & -2\Delta_3[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)] \\ & +(\Delta+\Delta_3)[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)] \\ & +\ \Delta_3[G(\mathbf{Z}_3,\mathbf{Z}_1)-G(\mathbf{Z}_1,\mathbf{Z}_3)]+\mathcal{C}, \end{aligned}$$

where $\mathcal{C}$ is (another) arbitrary constant vector.

Example 4.14. The general solution of the equation

$$\begin{aligned} & 3[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ = \ & (3-\omega)f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+(3+2\omega)f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-\omega f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = -2\omega F(\mathbf{Z}_1,\mathbf{Z}_2)+(3+\omega)F(\mathbf{Z}_2,\mathbf{Z}_3)+\omega F(\mathbf{Z}_3,\mathbf{Z}_1)+\mathcal{C},$$

where $F:\ \mathcal{V}^2 \mapsto \mathcal{V}$ is an arbitrary function and $\mathcal{C} \in \mathcal{V}$ is an arbitrary constant vector.

For

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \alpha_2 \end{vmatrix} \neq 0,$$

from (4.5) we obtain

$$(\Delta-\Delta_2)F(\mathbf{Z}_1,\mathbf{Z}_2)-\Delta_3 F(\mathbf{Z}_2,\mathbf{Z}_1)-\Delta_1 F(\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}. \tag{4.28}$$

First we suppose that $\Delta-\Delta_1-\Delta_2-\Delta_3 \neq 0$. In this case by the substitution $\mathbf{Z}_2=\mathbf{Z}_1$ the equation (4.28) reduces to $F(\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$. On the basis of the last equality, the equation (4.28) becomes

$$(\Delta-\Delta_2)F(\mathbf{Z}_1,\mathbf{Z}_2)-\Delta_3 F(\mathbf{Z}_2,\mathbf{Z}_1)=\mathbf{O}.$$

By the permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$, from the above equation it follows that

$$-\Delta_3 F(\mathbf{Z}_1,\mathbf{Z}_2)+(\Delta-\Delta_2)F(\mathbf{Z}_2,\mathbf{Z}_1)=\mathbf{O}.$$

The system of the last two equations has a nontrivial solution if and only if the following condition $(\Delta-\Delta_2)^2=\Delta_3^2$ is satisfied.

Let $\Delta-\Delta_2=\Delta_3$ ($\neq 0$), then we obtain

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1),$$

where G satisfies $G(\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$.

Example 4.15. Every solution of the functional equation

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)=f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)$$

is given by the function

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1) + 3G(\mathbf{Z}_2,\mathbf{Z}_3)\\ + 3G(\mathbf{Z}_3,\mathbf{Z}_2) - G(\mathbf{Z}_3,\mathbf{Z}_1) - G(\mathbf{Z}_1,\mathbf{Z}_3),\end{aligned}$$

where G is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$ such that $G(\mathbf{Z}_1,\mathbf{Z}_1) \equiv \mathbf{O}$.

For the case $\Delta - \Delta_2 = -\Delta_3$ ($\neq 0$) the general solution is

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = G(\mathbf{Z}_1,\mathbf{Z}_2) - G(\mathbf{Z}_2,\mathbf{Z}_1).$$

Example 4.16. The functional equation

$$f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) = -f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) - f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)$$

has the function

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = G(\mathbf{Z}_3,\mathbf{Z}_1) - G(\mathbf{Z}_1,\mathbf{Z}_3)$$

as the general solution, where G is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$.

For $\Delta - \Delta_2 = \Delta_3 = 0$, the unique condition which must be satisfied by the function F is $F(\mathbf{Z}_1,\mathbf{Z}_1) \equiv \mathbf{O}$.

Example 4.17. The function

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = F(\mathbf{Z}_1,\mathbf{Z}_2) + F(\mathbf{Z}_2,\mathbf{Z}_3) \qquad (F(\mathbf{Z}_1,\mathbf{Z}_1) \equiv \mathbf{O})$$

is the general solution of the following equation

$$f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) = f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1).$$

The condition $(\Delta - \Delta_2)^2 \neq \Delta_3^2$ gives $F(\mathbf{Z}_1,\mathbf{Z}_2) \equiv \mathbf{O}$.

Example 4.18. The functional equation

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) = f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)$$

has the general solution $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}$.

Now we will pass on to the case $\Delta - \Delta_1 - \Delta_2 - \Delta_3 = 0$. Most of the arguments after the equality (4.23) apply to this case, so we will just give the results.

If $\Delta_1(\Delta_1 + 2\Delta_3) \neq 0$,

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = \frac{(\Delta_1+\Delta_3)R(\mathbf{Z}_1) + \Delta_3 R(\mathbf{Z}_2)}{\Delta_1 + 2\Delta_3},$$

with the notation $R(\mathbf{Z}_1) = F(\mathbf{Z}_1,\mathbf{Z}_1)$.

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Example 4.19. Every solution of the following functional equation

$$-2f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) = -f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1)$$

is given by the function

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 2R(\mathbf{Z}_1) + R(\mathbf{Z}_2) + 2R(\mathbf{Z}_3),$$

where R is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

For $\Delta_1 = 0,\ \Delta_3 \neq 0$ the solution of equation (4.28) is

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = G(\mathbf{Z}_1, \mathbf{Z}_2) + G(\mathbf{Z}_2, \mathbf{Z}_1).$$

Example 4.20. The general solution of the functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) = 2f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1)$$

is

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 2G(\mathbf{Z}_1, \mathbf{Z}_3) + 2G(\mathbf{Z}_3, \mathbf{Z}_1) - G(\mathbf{Z}_2, \mathbf{Z}_3) - G(\mathbf{Z}_3, \mathbf{Z}_2),$$

where G is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$.

Let $\Delta_1 = -\,2\Delta_3 \neq 0$. Then from equation (4.28) we get

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = G(\mathbf{Z}_1, \mathbf{Z}_2) - G(\mathbf{Z}_2, \mathbf{Z}_1) + \mathcal{C},$$

where $\mathcal{C}$ is a constant complex vector.

Example 4.21. The general solution of the functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + 3f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1)$$

is determined by the function

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 2G(\mathbf{Z}_1, \mathbf{Z}_2) - 2G(\mathbf{Z}_2, \mathbf{Z}_1) + G(\mathbf{Z}_1, \mathbf{Z}_3) - G(\mathbf{Z}_3, \mathbf{Z}_1) + \mathcal{C},$$

where G is an arbitrary function $\mathcal{V}^2 \mapsto \mathcal{V}$ and $\mathcal{C} \in \mathcal{V}$ is an arbitrary constant vector.

In the case $\Delta_1 = \Delta_3 = 0$ the equation (4.28) is satisfied for every function $F:\ \mathcal{V}^2 \mapsto \mathcal{V}$. In this case by Lemma 4.1 we again obtain $\alpha_1 = a_3,\ \alpha_2 = a_1,\ \alpha_3 = a_2$. So the equation

$$\begin{aligned} & a_1 f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + a_2 f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + a_3 f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \; & a_3 f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + a_1 f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + a_2 f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

has the general solution $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = F(\mathbf{Z}_2, \mathbf{Z}_3)$.

Example 4.22. The general solution of the functional equation

$$\begin{aligned} & 2f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + 3f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \; & f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + 2f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + 3f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

is

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=F(\mathbf{Z}_2,\mathbf{Z}_3),$$

where F is an arbitrary function $\mathcal{V}^2\mapsto\mathcal{V}$.

2°. Let $\Delta=0$. Then from (4.4) it follows that

$$a_1+a_2+a_3=0 \qquad \text{or} \qquad (a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2=0.$$

First we will consider the case $a_1=a_2=a_3$. From (4.1) and (4.2) we obtain

$$(\alpha_1-\alpha_3)F(\mathbf{Z}_1,\mathbf{Z}_2)+(\alpha_2-\alpha_1)F(\mathbf{Z}_2,\mathbf{Z}_3)+(\alpha_3-\alpha_2)F(\mathbf{Z}_3,\mathbf{Z}_1)=\mathbf{O} \tag{4.29}$$

with the notation $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)=F(\mathbf{Z}_1,\mathbf{Z}_2)$. If $\alpha_1=\alpha_2=\alpha_3$, then the condition (4.29) is satisfied for every function F. For the case $\alpha_1=\alpha_2=\alpha_3\ (\neq 0)$, equation (4.1) takes the form

$$\begin{aligned} & a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\alpha_1f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)\\ +\ & a_1f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-\alpha_1f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)\\ +\ & a_1f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)-\alpha_1f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)=\mathbf{O}. \end{aligned} \tag{4.30}$$

This quasicyclic equation has the general solution

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\frac{\alpha_1}{a_1}F(\mathbf{Z}_1,\mathbf{Z}_2)+U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) \tag{4.31}$$

with the notation $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)=F(\mathbf{Z}_1,\mathbf{Z}_2)$.

By the substitution of (4.31) into (4.30) we obtain

$$\begin{aligned} & F(\mathbf{Z}_1,\mathbf{Z}_2)-\frac{\alpha_1}{a_1}F(\mathbf{Z}_1,\mathbf{Z}_1)-U(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)\\ +\ & F(\mathbf{Z}_2,\mathbf{Z}_3)-\frac{\alpha_1}{a_1}F(\mathbf{Z}_2,\mathbf{Z}_2)-U(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_2)\\ +\ & F(\mathbf{Z}_3,\mathbf{Z}_1)-\frac{\alpha_1}{a_1}F(\mathbf{Z}_3,\mathbf{Z}_3)-U(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_3)=\mathbf{O}. \end{aligned}$$

This quasicyclic equation has the general solution

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=\frac{\alpha_1}{a_1}F(\mathbf{Z}_1,\mathbf{Z}_1)+U(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+R(\mathbf{Z}_1)-R(\mathbf{Z}_2),$$

where R is an arbitrary function $\mathcal{V}\mapsto\mathcal{V}$.

By using the last equality, for $\alpha_1=a_1$, the equality (4.31) becomes

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)\\ &+\ U(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+S(\mathbf{Z}_1)-R(\mathbf{Z}_2), \end{aligned} \tag{4.32}$$

where $S:\ \mathcal{V}\mapsto\mathcal{V}$ is such that $F(\mathbf{Z}_1,\mathbf{Z}_1)=S(\mathbf{Z}_1)-R(\mathbf{Z}_1)$.

Example 4.23. Every solution of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ =\ & f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the function (4.32).

For $\alpha_1 \neq a_1$, from (4.1) there follows that $F(\mathbf{Z}_1, \mathbf{Z}_1) \equiv \mathbf{O}$. According to the last identity, the equality (4.31) is transformed into

$$\begin{aligned} & f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) \\ + \quad & \frac{\alpha_1}{a_1}[U(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1)] + R(\mathbf{Z}_1) - R(\mathbf{Z}_2). \end{aligned}$$

Example 4.24. The functional equation

$$\begin{aligned} & 2f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + 2f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + 2f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \quad & f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1) \end{aligned}$$

has the general solution

$$\begin{aligned} & f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) \\ + \quad & \frac{1}{2}[U(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1)] + R(\mathbf{Z}_1) - R(\mathbf{Z}_2). \end{aligned}$$

Now we will suppose that the parameters α_i $(1 \leq i \leq 3)$ are not all equal. Let $\alpha_1 \neq \alpha_3$. According to the equality (4.29) for $\mathbf{Z}_3 = \mathbf{A}$ (a constant complex vector) we obtain

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = K(\mathbf{Z}_1) + H(\mathbf{Z}_2), \tag{4.33}$$

where we used the notations

$$K(\mathbf{Z}_1) = -\frac{\alpha_3 - \alpha_2}{\alpha_1 - \alpha_3} F(\mathbf{A}, \mathbf{Z}_1), \quad H(\mathbf{Z}_2) = -\frac{\alpha_2 - \alpha_1}{\alpha_1 - \alpha_3} F(\mathbf{Z}_2, \mathbf{A}).$$

If we substitute $F(\mathbf{Z}_1, \mathbf{Z}_2)$ given by the expression (4.33) into (4.29), and if we set $\mathbf{Z}_1 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{B}$ (a constant complex vector) and if, on the other hand, we set $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{B}$, $\mathbf{Z}_2 = \mathbf{U}$, we obtain respectively

$$(\alpha_1 - \alpha_3)[K(\mathbf{U}) - K(\mathbf{B})] + (\alpha_3 - \alpha_2)[H(\mathbf{U}) - H(\mathbf{B})] = \mathbf{O}, \tag{4.34}$$

$$(\alpha_2 - \alpha_1)[K(\mathbf{U}) - K(\mathbf{B})] + (\alpha_1 - \alpha_3)[H(\mathbf{U}) - H(\mathbf{B})] = \mathbf{O}. \tag{4.35}$$

The determinant of this system is

$$\begin{vmatrix} \alpha_1 - \alpha_3 & \alpha_3 - \alpha_2 \\ \alpha_2 - \alpha_1 & \alpha_1 - \alpha_3 \end{vmatrix} = \frac{1}{2}[(\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_1)^2].$$

If it is not 0, then from (4.34) and (4.35) we find $K(\mathbf{U}) = K(\mathbf{B})$ and $H(\mathbf{U}) = H(\mathbf{B})$, hence

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{M} \qquad \text{(a constant complex vector)}. \tag{4.36}$$

Now the equation (4.1) becomes

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - \mathbf{N} + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) - \mathbf{N} + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) - \mathbf{N} = \mathbf{O}, \tag{4.37}$$

where

$$\mathbf{N} = \frac{\alpha_1 + \alpha_2 + \alpha_3}{3a_1}\mathbf{M}.$$

The general solution of the cyclic functional equation (4.37) is

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - p(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + \mathbf{N}. \tag{4.38}$$

From (4.36) and (4.38) we find

$$\mathbf{M} = F(\mathbf{Z}_1, \mathbf{Z}_2) = p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1) + \mathbf{N}.$$

If we put into the last equality $\mathbf{Z}_2 = \mathbf{Z}_1$, then $\mathbf{M} = \mathbf{N}$. This is possible if

$$\alpha_1 + \alpha_2 + \alpha_3 = 3a_1 \qquad \text{or} \qquad \mathbf{N} = \mathbf{O}.$$

Moreover,

$$p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1) = \mathbf{O}. \tag{4.39}$$

Now we shall use

Lemma 4.2. *Let* $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ *be a function of the form*

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - p(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1)$$

such that $f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}$. *Then*

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = {} & U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) \\ & - U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + U(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2), \end{aligned} \tag{4.40}$$

where $U : \mathcal{V}^3 \mapsto \mathcal{V}$ *is an arbitrary function.*

Proof. Let $p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ satisfy equation (4.39). We are looking for $p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ in the form

$$\begin{aligned} p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= k_1 q(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + k_2 q(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) + k_3 q(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) \\ &+ k_4 q(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + k_5 q(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) + k_6 q(\mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_1), \end{aligned}$$

where $q : \mathcal{V}^3 \mapsto \mathcal{V}$ and k_i $(1 \le i \le 6)$ are complex constants. By a substitution into (4.39) we find

$$k_5 = k_1 - k_2 + k_3, \quad k_6 = k_4 - k_1 + k_2.$$

Thus

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = {} & k[q(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) - q(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)] \\ & - \ell[q(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) - q(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1)] \\ & + (\ell - k)[q(\mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_1) - q(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2)], \end{aligned}$$

where k, ℓ are complex constants ($k = k_3 - k_2$, $\ell = k_4 - k_1$).

If we denote

$$U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = \ell q(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + k q(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2),$$

we obtain (4.40). Conversely, each function of the form (4.40) satisfies $f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}$ for arbitrary $U : \mathcal{V}^3 \mapsto \mathcal{V}$. Moreover, $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ satisfies

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}.$$

We note that the representation (4.40) can be obtained just by putting

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3). \qquad \square$$

Thus the general solution of equation (4.1) in this case is given by

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) \\ &\quad - U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + U(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) + \mathbf{N}, \end{aligned} \tag{4.41}$$

where U is an arbitrary function $\mathcal{V}^3 \mapsto \mathcal{V}$ and $\mathbf{N}$ is a constant vector in $\mathcal{V}$, $\mathbf{N} = \mathbf{O}$ if $\alpha_1 + \alpha_2 + \alpha_3 \neq 3a_1$.

Example 4.25. The function (4.40) is the general solution of the functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3).$$

Example 4.26. The function (4.41) is the general solution of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \; & f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + 3f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1). \end{aligned}$$

If the determinant of the system (4.34), (4.35) is 0, then α_1, α_2, α_3 are distinct complex constants related by the equality (4.21). In this case the relation (4.33) can be written in the form

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = (\alpha_3 - \alpha_1)K(\mathbf{Z}_1) + (\alpha_2 - \alpha_1)K(\mathbf{Z}_2),$$

where $G : \mathcal{V} \mapsto \mathcal{V}$ is an arbitrary function. Then (4.1) takes on the form

$$\begin{aligned} & a_1[f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2)] \\ & - (\alpha_1^2 - \alpha_2\alpha_3)[K(\mathbf{Z}_1) + K(\mathbf{Z}_2) + K(\mathbf{Z}_3)] = \mathbf{O}. \end{aligned}$$

If

$$\alpha_1^2 = \alpha_2\alpha_3, \tag{4.42}$$

then the functional equation becomes

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}$$

and its general solution is

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - p(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1).$$

We have

$$\begin{aligned} F(\mathbf{Z}_1,\mathbf{Z}_2) &= p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) - p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) \\ &= (\alpha_3-\alpha_1)K(\mathbf{Z}_1) + (\alpha_2-\alpha_1)K(\mathbf{Z}_2). \end{aligned}$$

For $\mathbf{Z}_2 = \mathbf{Z}_1$ we obtain

$$\mathbf{O} \equiv (\alpha_2+\alpha_3-2\alpha_1)K(\mathbf{Z}_1). \tag{4.43}$$

If we suppose that

$$\alpha_1 = \frac{\alpha_2+\alpha_3}{2},$$

this equality, together with (4.42), implies that $\alpha_1 = \alpha_2 = \alpha_3$ which is a contradiction. So $\alpha_2+\alpha_3-2\alpha_1 \neq 0$ and (4.43) implies that $K(\mathbf{Z}_1) \equiv \mathbf{O}$, *i.e.*, $F(\mathbf{Z}_1,\mathbf{Z}_2) \equiv \mathbf{O}$. By virtue of Lemma 4.2 the general solution of equation (4.1) is given by (4.40), where $U: \mathcal{V}^3 \mapsto \mathcal{V}$ is an arbitrary function.

Example 4.27. The general solution of the equation

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = \; - \; & f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \frac{1+i\sqrt{3}}{2} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) + \frac{1-i\sqrt{3}}{2} f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by (4.40), where U is an arbitrary function.

Now we suppose that the condition (4.42) is not satisfied. The function

$$g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \frac{\alpha_1^2-\alpha_2\alpha_3}{a_1} K(\mathbf{Z}_1)$$

satisfies the cyclic equation

$$g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + g(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) = \mathbf{O}. \tag{4.44}$$

Moreover,

$$\begin{aligned} g(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) &= F(\mathbf{Z}_1,\mathbf{Z}_2) - \frac{\alpha_1^2-\alpha_2\alpha_3}{a_1} K(\mathbf{Z}_1) \\ &= (\alpha_3-\alpha_1)K(\mathbf{Z}_1) + (\alpha_2-\alpha_1)K(\mathbf{Z}_2) - \frac{\alpha_1^2-\alpha_2\alpha_3}{a_1} K(\mathbf{Z}_1). \end{aligned}$$

Since $g(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1) \equiv \mathbf{O}$, we obtain

$$\left(\alpha_2+\alpha_3-2\alpha_1 - \frac{\alpha_1^2-\alpha_2\alpha_3}{a_1}\right) K(\mathbf{Z}_1) \equiv \mathbf{O}. \tag{4.45}$$

First let us suppose that

$$\frac{\alpha_1^2-\alpha_2\alpha_3}{a_1} = \alpha_2+\alpha_3-2\alpha_1. \tag{4.46}$$

Then (4.45) is satisfied for an arbitrary function $K:\ \mathcal{V}\mapsto\mathcal{V}$. Now we have

$$g(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)=(\alpha_2-\alpha_1)[K(\mathbf{Z}_2)-K(\mathbf{Z}_1)]. \tag{4.47}$$

If $H:\ \mathcal{V}\mapsto\mathcal{V}$ is an arbitrary function, then

$$g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=H(\mathbf{Z}_1)-H(\mathbf{Z}_2)+(\alpha_2-\alpha_1)[K(\mathbf{Z}_3)-K(\mathbf{Z}_1)]$$

is a solution of (4.44) satisfying (4.47).

The general solution of (4.44) satisfying (4.47) is

$$\begin{aligned}g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=&\,U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)\\&+U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)+\ H(\mathbf{Z}_1)-H(\mathbf{Z}_2)\\&+(\alpha_2-\alpha_1)[K(\mathbf{Z}_3)-K(\mathbf{Z}_1)],\end{aligned}$$

where $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function.

We note that

$$\begin{aligned}&U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)\\&\qquad+H(\mathbf{Z}_1)-H(\mathbf{Z}_2)\\&=\ \tilde{U}(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\tilde{U}(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-\tilde{U}(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\tilde{U}(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2),\end{aligned}$$

where

$$\tilde{U}(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\frac{1}{2}H(\mathbf{Z}_1).$$

Thus the general solution of (4.1) in this case is

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=&\,U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)\\&+U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)+\ (\alpha_3-\alpha_1)K(\mathbf{Z}_1)+(\alpha_2-\alpha_1)K(\mathbf{Z}_3).\end{aligned}$$

Example 4.28. The general solution of the equation

$$\begin{aligned}&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\&=\ \ -i\sqrt{3}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+i\sqrt{3}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-3f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=&\,U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)\\&+U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)+\ (i\sqrt{3}-3)K(\mathbf{Z}_1)+2i\sqrt{3}K(\mathbf{Z}_3),\end{aligned}$$

where $K:\ \mathcal{V}\mapsto\mathcal{V}$ and $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions.

Now we suppose that (4.46) is not satisfied. Then (4.45) implies $K(\mathbf{Z}_1)\equiv\mathbf{O}$. In this case (4.40) is the general solution of equation (4.1).

Example 4.29. The general solution of the equation

$$\begin{aligned}&2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+2f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\&=\ \ -i\sqrt{3}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+i\sqrt{3}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-3f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3) \\ &\quad - U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2), \end{aligned}$$

where $U:\ \mathcal{V}^3 \mapsto \mathcal{V}$ is an arbitrary function.

Now we will consider the case

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 \neq 0. \tag{4.48}$$

Thus we have

$$a_1 + a_2 + a_3 = 0. \tag{4.49}$$

If we suppose that $a_1^3 = a_2^3 = a_3^3$, by virtue of (4.49) we obtain

$$a_1^2 = a_2 a_3, \quad a_2^2 = a_1 a_3, \quad a_3^2 = a_1 a_2$$

which contradicts (4.48). So we may suppose that, for instance, $a_1^3 \neq a_2^3$ (in particular, $a_1 \neq a_2$). The equation (4.1) can be written as

$$\begin{aligned} &a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] - a_2[f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) - f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)] \\ &\quad = \alpha_1 F(\mathbf{Z}_1,\mathbf{Z}_2) + \alpha_2 F(\mathbf{Z}_2,\mathbf{Z}_3) + \alpha_3 F(\mathbf{Z}_3,\mathbf{Z}_1), \end{aligned} \tag{4.50}$$

where $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) = F(\mathbf{Z}_1,\mathbf{Z}_2)$. Also from (4.2) and (4.3) it follows that

$$\begin{aligned} &a_1[f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)] - a_2[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ &\quad = \alpha_1 F(\mathbf{Z}_2,\mathbf{Z}_3) + \alpha_2 F(\mathbf{Z}_3,\mathbf{Z}_1) + \alpha_3 F(\mathbf{Z}_1,\mathbf{Z}_2), \end{aligned} \tag{4.51}$$

$$\begin{aligned} &a_1[f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) - f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)] - a_2[f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)] \\ &\quad = \alpha_1 F(\mathbf{Z}_3,\mathbf{Z}_1) + \alpha_2 F(\mathbf{Z}_1,\mathbf{Z}_2) + \alpha_3 F(\mathbf{Z}_2,\mathbf{Z}_3). \end{aligned} \tag{4.52}$$

By addition of (4.50), (4.51) and (4.52) we obtain

$$(\alpha_1 + \alpha_2 + \alpha_3)[F(\mathbf{Z}_1,\mathbf{Z}_2) + F(\mathbf{Z}_2,\mathbf{Z}_3) + F(\mathbf{Z}_3,\mathbf{Z}_1)] = \mathbf{O}.$$

For $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$, the following condition must be satisfied

$$F(\mathbf{Z}_1,\mathbf{Z}_2) + F(\mathbf{Z}_2,\mathbf{Z}_3) + F(\mathbf{Z}_3,\mathbf{Z}_1) = \mathbf{O}.$$

This cyclic functional equation has the general solution

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = P(\mathbf{Z}_1) - P(\mathbf{Z}_2), \tag{4.53}$$

where P is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

From the equations (4.50), (4.51) and (4.52), if we take into account (4.53), we get

$$\begin{aligned}
&f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&+\frac{1}{a_1^3-a_2^3}\left\{a_1^2[\alpha_1P(\mathbf{Z}_1)+\alpha_2P(\mathbf{Z}_2)+\alpha_3P(\mathbf{Z}_3)]\right.\\
&+a_2^2[\alpha_3P(\mathbf{Z}_1)+\alpha_1P(\mathbf{Z}_2)+\alpha_2P(\mathbf{Z}_3)]\\
&\left.+a_1a_2[\alpha_2P(\mathbf{Z}_1)+\alpha_3P(\mathbf{Z}_2)+\alpha_1P(\mathbf{Z}_3)]\right\}\\
&=f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\frac{1}{a_1^3-a_2^3}\left\{a_1^2[\alpha_1P(\mathbf{Z}_2)+\alpha_2P(\mathbf{Z}_3)+\alpha_3P(\mathbf{Z}_1)]\right.\\
&+a_2^2[\alpha_3P(\mathbf{Z}_2)+\alpha_1P(\mathbf{Z}_3)+\alpha_2P(\mathbf{Z}_1)]\\
&\left.+a_1a_2[\alpha_2P(\mathbf{Z}_2)+\alpha_3P(\mathbf{Z}_3)+\alpha_1P(\mathbf{Z}_1)]\right\}.
\end{aligned}$$

The last equation has the general solution

$$\begin{aligned}
(4.54)\qquad &f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\frac{1}{a_1^3-a_2^3}\left\{a_1^2[\alpha_1P(\mathbf{Z}_2)+\alpha_2P(\mathbf{Z}_3)+\alpha_3P(\mathbf{Z}_1)]\right.\\
&+a_2^2[\alpha_3P(\mathbf{Z}_2)+\alpha_1P(\mathbf{Z}_3)+\alpha_2P(\mathbf{Z}_1)]\\
&\left.+a_1a_2[\alpha_2P(\mathbf{Z}_2)+\alpha_3P(\mathbf{Z}_3)+\alpha_1P(\mathbf{Z}_1)]\right\}\\
&=p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}$$

where p is an arbitrary function $\mathcal{V}^3\mapsto\mathcal{V}$.

By virtue of (4.53) $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)=P(\mathbf{Z}_1)-P(\mathbf{Z}_2)$, then from (4.54) it follows that

$$\begin{aligned}
(4.55)\qquad &P(\mathbf{Z}_1)-P(\mathbf{Z}_2)+\frac{1}{a_1^3-a_2^3}\left\{a_1^2[(\alpha_1+\alpha_3)P(\mathbf{Z}_1)+\alpha_2P(\mathbf{Z}_2)]\right.\\
+\quad &\left.a_2^2[(\alpha_2+\alpha_3)P(\mathbf{Z}_1)+\alpha_1P(\mathbf{Z}_2)]+a_1a_2[(\alpha_1+\alpha_2)P(\mathbf{Z}_1)+\alpha_3P(\mathbf{Z}_2)]\right\}\\
&=p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).
\end{aligned}$$

For $\mathbf{Z}_2=\mathbf{Z}_1$ this equality takes the form

$$\frac{\alpha_1+\alpha_2+\alpha_3}{a_1^3-a_2^3}(a_1^2+a_2^2+a_1a_2)P(\mathbf{Z}_1)=3p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1),$$

which implies

$$P(\mathbf{Z}_1)=\frac{3(a_1-a_2)}{\alpha_1+\alpha_2+\alpha_3}p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1).$$

Now from (4.54) we find the general solution in the form

$$\begin{aligned}
(4.56)\qquad f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&\quad -\frac{3}{(a_1^2+a_2^2+a_1a_2)(\alpha_1+\alpha_2+\alpha_3)}\\
&\quad\times \big\{a_1^2[\alpha_1 p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_2 p(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_3)+\alpha_3 p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)]\\
&\quad + a_2^2[\alpha_1 p(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_3)+\alpha_2 p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_3 p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)]\\
&\quad + a_1a_2[\alpha_1 p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_2 p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_3 p(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_3)]\big\}\,,
\end{aligned}$$

where $p:\ \mathcal{V}^3\mapsto\mathcal{V}$ must satisfy the following condition derived from (4.55):

$$\begin{aligned}
(4.57)\qquad &\frac{3(a_1-a_2)}{\alpha_1+\alpha_2+\alpha_3}[p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)-p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)]\\
&+\frac{3}{(a_1^2+a_2^2+a_1a_2)(\alpha_1+\alpha_2+\alpha_3)}\big\{a_1^2[(\alpha_1+\alpha_3)p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)\\
&\quad+\alpha_2 p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)]\\
&\quad+a_2^2[(\alpha_2+\alpha_3)p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_1 p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)]\\
&\quad+a_1a_2[(\alpha_1+\alpha_2)p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_3 p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)]\big\}\\
&\quad=p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).
\end{aligned}$$

It is easily seen that (4.57) is an equation of the form

$$\begin{aligned}
(4.58)\qquad & p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)\\
&=(3-\gamma)p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\gamma p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2),
\end{aligned}$$

where the complex constant γ is given by

$$\gamma=-\frac{3(a_1-a_2)}{\alpha_1+\alpha_2+\alpha_3}+\frac{3(a_1^2\alpha_2+a_2^2\alpha_1+a_1a_2\alpha_3)}{(a_1^2+a_2^2+a_1a_2)(\alpha_1+\alpha_2+\alpha_3)}.$$

Lemma 4.3. *Let $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)$ be a function of the form*

$$(4.59)\qquad f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)$$

such that $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)=\mathbf{O}$. Then

$$\begin{aligned}
(4.60)\qquad f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)= &\ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),
\end{aligned}$$

where $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function.

Proof. We are looking for a function of the form

$$\begin{aligned}
p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \ =\ & k_1q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+k_2q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+k_3q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&+k_4q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)+k_5q(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)+k_6q(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),
\end{aligned}$$

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where k_i $(1 \leq i \leq 6)$ are complex constants, satisfying

$$p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1) + p(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_1) = \mathbf{O} \tag{4.61}$$

for any function $q: \mathcal{V}^3 \mapsto \mathcal{V}$. By a substitution into (4.61) we find

$$k_1 + k_2 + k_3 = k_4 + k_5 + k_6.$$

Thus

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = {} & (k_1 + k_2 + k_3)[q(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + q(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + q(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2)] \\ & - (k_1 + k_2 + k_3)[q(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) + q(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) + q(\mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_1)]. \end{aligned}$$

If we put

$$U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = (k_1 + k_2 + k_3) q(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3),$$

we obtain the representation (4.60).

We note that the representation (4.60) can be obtained from (4.59) by putting

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3). \qquad \square$$

Let us suppose that $p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_1) = S(\mathbf{Z}_1) \not\equiv \mathbf{O}$. The equation

$$p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1) + p(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_1) = (3 - \gamma) S(\mathbf{Z}_1) + \gamma S(\mathbf{Z}_2)$$

has a nonconstant solution of the form

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = S(\mathbf{Z}_1)$$

or, more generally,

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = m_1 S(\mathbf{Z}_1) + m_2 S(\mathbf{Z}_2) + (1 - m_1 - m_2) S(\mathbf{Z}_3)$$

only if $\gamma = 1$. Indeed, we have

$$(\gamma - 1)[S(\mathbf{Z}_1) - S(\mathbf{Z}_2)] = \mathbf{O}.$$

On the other hand, any $S(\mathbf{Z}_1) \equiv \mathbf{A}$, where $\mathbf{A}$ is a constant vector, satisfies the last equality.

Let us put

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = \tilde{p}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + S(\mathbf{Z}_1).$$

Then $\tilde{p}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ satisfies an equation of the form (4.61) and we have proved

Corollary 4.4. *Let $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ be a function of the form* (4.59) *such that $p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ satisfies* (4.58). *Then*

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \quad = \quad & U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + U(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ & - U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) - U(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) - U(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ & + S(\mathbf{Z}_1) + S(\mathbf{Z}_2) + S(\mathbf{Z}_3), \end{aligned}$$

where $U: \mathcal{V}^3 \mapsto \mathcal{V}$ is an arbitrary function and S is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$ for $\gamma = 1$, S is equal to a constant vector $\mathbf{A} \in \mathcal{V}$ otherwise.

Thus from (4.56) we find that $f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ is given by (4.60) if $\gamma \neq 1$,

$$\begin{aligned}
&f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = S(\mathbf{Z}_1) + S(\mathbf{Z}_2) + S(\mathbf{Z}_3) \\
&- \frac{3}{(a_1^2 + a_2^2 + a_1 a_2)(\alpha_1 + \alpha_2 + \alpha_3)} \left\{ a_1^2[\alpha_3 S(\mathbf{Z}_1) + \alpha_1 S(\mathbf{Z}_2) + \alpha_2 S(\mathbf{Z}_3)] \right. \\
&+ a_2^2[\alpha_2 S(\mathbf{Z}_1) + \alpha_3 S(\mathbf{Z}_2) + \alpha_1 S(\mathbf{Z}_3)] \\
&\left. + a_1 a_2[\alpha_1 S(\mathbf{Z}_1) + \alpha_2 S(\mathbf{Z}_2) + \alpha_3 S(\mathbf{Z}_3)] \right\} \\
&+ U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + U(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\
&- U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) - U(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) - U(\mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_1).
\end{aligned}$$

Example 4.30. The general solution of the functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) = f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2)$$

is given by (4.60).

Example 4.31. The functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) = 3f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2)$$

has the general solution

$$\begin{aligned}
&f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 2S(\mathbf{Z}_1) - S(\mathbf{Z}_2) - S(\mathbf{Z}_3) \\
+ \quad &U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + U(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\
- \quad &U(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_3) - U(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_2) - U(\mathbf{Z}_3, \mathbf{Z}_2, \mathbf{Z}_1),
\end{aligned}$$

where $S: \mathcal{V} \mapsto \mathcal{V}$ and $U: \mathcal{V}^3 \mapsto \mathcal{V}$ are arbitrary functions.

Now we pass on to the case $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Then from (4.50), (4.51) and (4.52) we obtain

$$\begin{aligned}
(4.62)\; &f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) - \frac{1}{a_1^3 - a_2^3} \left\{ a_1^2[\alpha_1 F(\mathbf{Z}_3, \mathbf{Z}_1) - \alpha_2 F(\mathbf{Z}_2, \mathbf{Z}_3)] \right. \\
+ \quad &\left. a_2^2[\alpha_1 F(\mathbf{Z}_1, \mathbf{Z}_2) - \alpha_2 F(\mathbf{Z}_3, \mathbf{Z}_1)] + a_1 a_2[\alpha_1 F(\mathbf{Z}_2, \mathbf{Z}_3) - \alpha_2 F(\mathbf{Z}_1, \mathbf{Z}_2)] \right\} \\
= \quad &f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - \frac{1}{a_1^3 - a_2^3} \left\{ a_1^2[\alpha_1 F(\mathbf{Z}_1, \mathbf{Z}_2) - \alpha_2 F(\mathbf{Z}_3, \mathbf{Z}_1)] \right. \\
+ \quad &\left. a_2^2[\alpha_1 F(\mathbf{Z}_2, \mathbf{Z}_3) - \alpha_2 F(\mathbf{Z}_1, \mathbf{Z}_2)] + a_1 a_2[\alpha_1 F(\mathbf{Z}_3, \mathbf{Z}_1) - \alpha_2 F(\mathbf{Z}_2, \mathbf{Z}_3)] \right\}.
\end{aligned}$$

The general solution of the equation (4.62) is given by the formula

$$\begin{aligned}
(4.63)\; &f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = \frac{1}{a_1^3 - a_2^3} \left\{ a_1^2[\alpha_1 F(\mathbf{Z}_1, \mathbf{Z}_2) - \alpha_2 F(\mathbf{Z}_3, \mathbf{Z}_1)] \right. \\
+ \quad &\left. a_2^2[\alpha_1 F(\mathbf{Z}_2, \mathbf{Z}_3) - \alpha_2 F(\mathbf{Z}_1, \mathbf{Z}_2)] + a_1 a_2[\alpha_1 F(\mathbf{Z}_3, \mathbf{Z}_1) - \alpha_2 F(\mathbf{Z}_2, \mathbf{Z}_3)] \right\} \\
&+ q(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + q(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + q(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2),
\end{aligned}$$

where $q: \mathcal{V}^3 \mapsto \mathcal{V}$ is an arbitrary function.

From the equality (4.63) we obtain

$$(4.64)\ F(\mathbf{Z}_1,\mathbf{Z}_2)=\frac{1}{a_1^3-a_2^3}\left\{a_1^2[\alpha_1F(\mathbf{Z}_1,\mathbf{Z}_1)-\alpha_2F(\mathbf{Z}_2,\mathbf{Z}_1)]\right.$$
$$+\quad \left.a_2^2[\alpha_1F(\mathbf{Z}_1,\mathbf{Z}_2)-\alpha_2F(\mathbf{Z}_1,\mathbf{Z}_1)]+a_1a_2[\alpha_1F(\mathbf{Z}_2,\mathbf{Z}_1)-\alpha_2F(\mathbf{Z}_1,\mathbf{Z}_2)]\right\}$$
$$+q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).$$

For $\mathbf{Z}_2=\mathbf{Z}_1$ (4.64) yields

$$(4.65)\qquad F(\mathbf{Z}_1,\mathbf{Z}_1)=\frac{\alpha_1-\alpha_2}{a_1-a_2}F(\mathbf{Z}_1,\mathbf{Z}_1)+3q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1).$$

If $\alpha_1-\alpha_2=a_1-a_2$, then $q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}$ and $F(\mathbf{Z}_1,\mathbf{Z}_1)=P(\mathbf{Z}_1)$, where P is an arbitrary function $\mathcal{V}\mapsto\mathcal{V}$. Now we have

(4.66)

$$\left[1+\frac{a_2(\alpha_1-a_1)}{a_1^2+a_2^2+a_1a_2}\right]F(\mathbf{Z}_1,\mathbf{Z}_2)+\frac{a_1(\alpha_1-a_1)}{a_1^2+a_2^2+a_1a_2}F(\mathbf{Z}_2,\mathbf{Z}_1)$$
$$=\frac{\alpha_1(a_1+a_2)+a_2^2}{a_1^2+a_2^2+a_1a_2}P(\mathbf{Z}_1)+q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).$$

First we consider the particular case $\alpha_1=a_1$ (then $\alpha_2=a_2$, $\alpha_3=a_3=-(a_1+a_2)$). Now

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=P(\mathbf{Z}_1)+q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).$$

Thus we find that the functional equation

$$\begin{aligned}&a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-(a_1+a_2)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ =\;&a_1f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+a_2f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-(a_1+a_2)f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

has the general solution

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\;=\;&P(\mathbf{Z}_1)+q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)\\&+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

where $P:\ \mathcal{V}\mapsto\mathcal{V}$ is an arbitrary function and $q:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function satisfying $q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$.

Now we consider equation (4.66) in the general case. By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we derive the equation

(4.67)

$$\frac{a_1(\alpha_1-a_1)}{a_1^2+a_2^2+a_1a_2}F(\mathbf{Z}_1,\mathbf{Z}_2)+\left[1+\frac{a_2(\alpha_1-a_1)}{a_1^2+a_2^2+a_1a_2}\right]F(\mathbf{Z}_2,\mathbf{Z}_1)$$
$$=\frac{\alpha_1(a_1+a_2)+a_2^2}{a_1^2+a_2^2+a_1a_2}P(\mathbf{Z}_2)+q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2).$$

The determinant of the system (4.66), (4.67) is

$$\frac{(a_2^2+a_1\alpha_1+a_2\alpha_1)(2a_1^2+a_2^2-a_1\alpha_1+a_2\alpha_1)}{(a_1^2+a_2^2+a_1a_2)^2}. \tag{4.68}$$

If this expression is not 0, then the solution of this system is

$$\begin{aligned} F(\mathbf{Z}_1,\mathbf{Z}_2) &= \frac{(a_1^2+a_2^2+a_2\alpha_1)P(\mathbf{Z}_1)-a_1(\alpha_1-a_1)P(\mathbf{Z}_2)}{2a_1^2+a_2^2-a_1\alpha_1+a_2\alpha_1} \\ &+ \frac{a_1^2+a_2^2+a_1a_2}{(a_2^2+a_1\alpha_1+a_2\alpha_1)(2a_1^2+a_2^2-a_1\alpha_1+a_2\alpha_1)} \\ &\times \{(a_1^2+a_2^2+a_2\alpha_1)[q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)] \\ &\quad -a_1(\alpha_1-a_1)[q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)]\}. \end{aligned}$$

Example 4.32. The general solution of the equation

$$\begin{aligned} &2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-3f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ &= f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{1}{7}[4F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_2,\mathbf{Z}_3)+2F(\mathbf{Z}_3,\mathbf{Z}_1)] \\ &\quad +q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2), \end{aligned}$$

where

$$\begin{aligned} F(\mathbf{Z}_1,\mathbf{Z}_2) &= \frac{1}{4}[3P(\mathbf{Z}_1)+P(\mathbf{Z}_2)] \\ &\quad +\frac{7}{16}\{3[q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)] \\ &\quad +q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)\}, \end{aligned}$$

P is an arbitrary function $\mathcal{V}\mapsto\mathcal{V}$ and q is an arbitrary function $\mathcal{V}^3\mapsto\mathcal{V}$ satisfying $q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$.

Now let us suppose that the expression (4.68) is 0. First let

$$a_2^2+\alpha_1(a_1+a_2)=0.$$

If $a_1+a_2=0$, then $a_1=a_2=a_3=0$ which is a contradiction. Thus

$$\alpha_1=-\frac{a_2^2}{a_1+a_2}, \qquad \alpha_2=-\frac{a_1^2}{a_1+a_2}.$$

Now (4.63) takes the form

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{a_1F(\mathbf{Z}_3,\mathbf{Z}_1)+a_2F(\mathbf{Z}_2,\mathbf{Z}_3)}{a_1+a_2} \\ &+ q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2), \end{aligned}$$

while (4.66) becomes

$$\begin{aligned}\frac{a_1}{a_1+a_2}[F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_1)] &= q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)\\ &+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned} \tag{4.69}$$

Equation (4.69) implies

$$q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}$$

and

$$F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_1)=\mathbf{O},$$

i.e.,

$$\begin{aligned} & q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \; & U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ - \; & U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),\end{aligned}$$

where $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function, and

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1),$$

where $G:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function.

Thus the general solution of the functional equation

$$\begin{aligned}& a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-(a_1+a_2)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &=-\frac{a_2^2}{a_1+a_2}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-\frac{a_1^2}{a_1+a_2}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\frac{a_1^2+a_2^2}{a_1+a_2}f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{a_1[G(\mathbf{Z}_1,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_1)]+a_2[G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2)]}{a_1+a_2}\\ &+ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &- U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1).\end{aligned}$$

Next we suppose that

$$2a_1^2+a_2^2-(a_1-a_2)\alpha_1=0.$$

Since $a_1\neq a_2$, we have

$$\alpha_1=\frac{2a_1^2+a_2^2}{a_1-a_2},\qquad \alpha_2=\frac{a_1^2+2a_1a_2}{a_1-a_2}.$$

Now (4.63) takes the form

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{1}{a_1-a_2}[2a_1F(\mathbf{Z}_1,\mathbf{Z}_2)-a_1F(\mathbf{Z}_3,\mathbf{Z}_1)-a_2F(\mathbf{Z}_2,\mathbf{Z}_3)]\\ &+ q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

while (4.66) becomes

$$\begin{aligned}(4.70)\qquad &\frac{a_1}{a_1-a_2}[F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_2,\mathbf{Z}_1)-2F(\mathbf{Z}_1,\mathbf{Z}_1)]\\ = \ & q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned}$$

Equation (4.70) implies

$$q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}$$

and

$$F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_2,\mathbf{Z}_1)-2F(\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O},$$

i.e.,

$$\begin{aligned}&q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ & U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ - \ & U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),\end{aligned}$$

where $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function, and

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)+\mathcal{C},$$

where $G:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function and $\mathcal{C}\in\mathcal{V}$ is an arbitrary constant vector.

Thus the general solution of the functional equation

$$\begin{aligned}&a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-(a_1+a_2)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ & \frac{2a_1^2+a_2^2}{a_1-a_2}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\frac{a_1^2+2a_1a_2}{a_1-a_2}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)\\ - \ & \frac{3a_1^2+2a_1a_2+a_2^2}{a_1-a_2}f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=&\frac{1}{a_1-a_2}\{2a_1[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)]\\ &+a_1[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)]\\ &-a_2[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)]\}+\mathcal{C}\\ &+U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1).\end{aligned}$$

Now let $\frac{\alpha_1-\alpha_2}{a_1-a_2}=\gamma\neq 1$. Then from (4.65) we find

$$(4.71)\qquad F(\mathbf{Z}_1,\mathbf{Z}_1)=\frac{3}{1-\gamma}q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1).$$

Now we have

$$\begin{aligned}(4.72)\qquad &\left[1+\frac{a_2(\alpha_1-a_1\gamma)}{a_1^2+a_2^2+a_1a_2}\right]F(\mathbf{Z}_1,\mathbf{Z}_2)+\frac{a_1(\alpha_1-a_1\gamma)}{a_1^2+a_2^2+a_1a_2}F(\mathbf{Z}_2,\mathbf{Z}_1)\\ &=\frac{3[\alpha_1(a_1+a_2)+a_2^2\gamma]}{(a_1^2+a_2^2+a_1a_2)(1-\gamma)}q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)\\ &+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned}$$

By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we derive the equation

$$\begin{aligned}(4.73)\qquad &\frac{a_1(\alpha_1-a_1\gamma)}{a_1^2+a_2^2+a_1a_2}F(\mathbf{Z}_1,\mathbf{Z}_2)+\left[1+\frac{a_2(\alpha_1-a_1\gamma)}{a_1^2+a_2^2+a_1a_2}\right]F(\mathbf{Z}_2,\mathbf{Z}_1)\\ &=\frac{3[\alpha_1(a_1+a_2)+a_2^2\gamma]}{(a_1^2+a_2^2+a_1a_2)(1-\gamma)}\,q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)+q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)\\ &+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2).\end{aligned}$$

The determinant of the system (4.72), (4.73) is

$$\begin{aligned}&\frac{[(a_1^2+a_1a_2)(1-\gamma)+a_2^2+(a_1+a_2)\alpha_1]}{(a_1^2+a_2^2+a_1a_2)^2}\\ (4.74)\qquad &\times[a_1^2(1+\gamma)+a_1a_2(1-\gamma)+a_2^2-(a_1-a_2)\alpha_1].\end{aligned}$$

If this expression is not 0, then

$$\begin{aligned}&F(\mathbf{Z}_1,\mathbf{Z}_2)\\ &=\frac{3[\alpha_1(a_1+a_2)+a_2^2\gamma]}{[(a_1^2+a_1a_2)(1-\gamma)+a_2^2+(a_1+a_2)\alpha_1]}\\ &\quad\times\frac{1}{[a_1^2(1+\gamma)+a_1a_2(1-\gamma)+a_2^2-(a_1-a_2)\alpha_1]}\\ &\quad\times\left\{[a_1^2+a_2^2+a_1a_2(1-\gamma)+a_2\alpha_1]q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)\right.\\ &\qquad\left.-a_1(\alpha_1-a_1\gamma)q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)\right\}\\ &+\frac{a_1^2+a_2^2+a_1a_2}{[(a_1^2+a_1a_2)(1-\gamma)+a_2^2+(a_1+a_2)\alpha_1]}\\ &\quad\times\frac{1}{[a_1^2(1+\gamma)+a_1a_2(1-\gamma)+a_2^2-(a_1-a_2)\alpha_1]}\\ &\quad\times\left\{[a_1^2+a_2^2+a_1a_2(1-\gamma)+a_2\alpha_1]\right.\\ &\qquad\times[q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)]\\ &\qquad\left.-a_1(\alpha_1-a_1\gamma)[q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)]\right\}.\end{aligned}$$

Example 4.33. The general solution of the equation

$$\begin{aligned}2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-3f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ =2f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-2f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{1}{7}[8F(\mathbf{Z}_1,\mathbf{Z}_2)+2F(\mathbf{Z}_2,\mathbf{Z}_3)+4F(\mathbf{Z}_3,\mathbf{Z}_1)]\\ &\quad +q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

where

$$\begin{aligned}F(\mathbf{Z}_1,\mathbf{Z}_2) &= \frac{8}{3}[5q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+4q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)]\\ &+ \frac{7}{9}\{5[q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)]\\ &\quad +4[q(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)]\}\end{aligned}$$

and q is an arbitrary function $\mathcal{V}^3\mapsto\mathcal{V}$.

Now let us suppose that the expression (4.74) is 0. First let

$$(a_1^2+a_1a_2)(1-\gamma)+a_2^2+(a_1+a_2)\alpha_1=0.$$

Then

$$\alpha_1=-\frac{a_2^2+(a_1^2+a_1a_2)(1-\gamma)}{a_1+a_2},\qquad \alpha_2=-\frac{a_1^2+(a_2^2+a_1a_2)(1-\gamma)}{a_1+a_2}.$$

Now (4.63) takes the form

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= (\gamma-1)F(\mathbf{Z}_1,\mathbf{Z}_2)+\frac{a_1F(\mathbf{Z}_3,\mathbf{Z}_1)+a_2F(\mathbf{Z}_2,\mathbf{Z}_3)}{a_1+a_2}\\ &+ q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

while (4.72) becomes

$$\begin{aligned}&\frac{a_1}{a_1+a_2}[F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_1)]\\ =\ &q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)-3q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned}$$

The last equation implies

$$\begin{aligned}&q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &- U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1)\\ &\quad +P(\mathbf{Z}_1)+P(\mathbf{Z}_2)+P(\mathbf{Z}_3),\\ F(\mathbf{Z}_1,\mathbf{Z}_2) &= G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1)-\frac{a_1+a_2}{a_1}P(\mathbf{Z}_1),\end{aligned}$$

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where $U:\ \mathcal{V}^3 \mapsto \mathcal{V},\ G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ and $P:\ \mathcal{V} \mapsto \mathcal{V}$ are arbitrary functions. The condition (4.71) yields

$$2G(\mathbf{Z}_1,\mathbf{Z}_1) = \left(\frac{a_1+a_2}{a_1} + \frac{3}{1-\gamma}\right) P(\mathbf{Z}_1).$$

Example 4.34. The general solution of the equation

$$\begin{aligned} & 6f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + 3f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - 9f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = \ & 5f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) - f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) - 4f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1) \\ &+ \frac{1}{3}\,[2G(\mathbf{Z}_1,\mathbf{Z}_1) + G(\mathbf{Z}_2,\mathbf{Z}_3) + G(\mathbf{Z}_3,\mathbf{Z}_2) \\ &\qquad -2G(\mathbf{Z}_2,\mathbf{Z}_2) + 2G(\mathbf{Z}_1,\mathbf{Z}_3) + 2G(\mathbf{Z}_3,\mathbf{Z}_1)] \\ &+ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ &- U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3) - U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2) - U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1), \end{aligned}$$

where $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ and $U:\ \mathcal{V}^3 \mapsto \mathcal{V}$ are arbitrary functions.

Next we suppose that

$$a_1^2(1+\gamma) + a_1a_2(1-\gamma) + a_2^2 - (a_1-a_2)\alpha_1 = 0.$$

In this case we have

$$\alpha_1 = \frac{a_2^2 + a_1^2(1+\gamma) + a_1a_2(1-\gamma)}{a_1-a_2}, \qquad \alpha_2 = \frac{a_1^2 + a_2^2(1-\gamma) + a_1a_2(1+\gamma)}{a_1-a_2}.$$

Now (4.63) takes the form

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \\ &= \frac{1}{a_1-a_2}\left\{[a_1(1+\gamma) + a_2(1-\gamma)]F(\mathbf{Z}_1,\mathbf{Z}_2) - a_1F(\mathbf{Z}_3,\mathbf{Z}_1) - a_2F(\mathbf{Z}_2,\mathbf{Z}_3)\right\} \\ &\quad + q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2), \end{aligned}$$

while (4.72) becomes

$$\begin{aligned} \frac{a_1}{a_1-a_2}[F(\mathbf{Z}_1,\mathbf{Z}_2) + F(\mathbf{Z}_2,\mathbf{Z}_1)] &= \frac{3}{a_1-a_2}\left(\frac{1+\gamma}{1-\gamma}a_1 + a_2\right) q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1) \\ &+ q(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1). \end{aligned}$$

If

$$\frac{3}{a_1-a_2}\left(\frac{1+\gamma}{1-\gamma}a_1 + a_2\right) \neq -1,$$

i.e.,

$$(2+\gamma)a_1 + (1-\gamma)a_2 \neq 0,$$

as above we find that

$$\begin{aligned} & q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = & \ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ - & \ U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1), \\ & F(\mathbf{Z}_1,\mathbf{Z}_2)=G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1) \end{aligned}$$

and

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = & \ \frac{1}{a_1-a_2}\left\{[a_1(1+\gamma)+a_2(1-\gamma)]\,[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)]\right) \\ & + a_1[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)]-a_2[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)]\} \\ & + U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ & - U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1), \end{aligned}$$

where $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ and $U:\ \mathcal{V}^3 \mapsto \mathcal{V}$ are arbitrary functions.

Example 4.35. The general solution of the equation

$$\begin{aligned} & 2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-3f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = & \ 11f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+9f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-20f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=5[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)] \\ + & \ 2[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)]-G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2) \\ + & \ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ - & \ U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1), \end{aligned}$$

where $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ and $U:\ \mathcal{V}^3 \mapsto \mathcal{V}$ are arbitrary functions.

If, however,

$$(2+\gamma)a_1+(1-\gamma)a_2=0,$$

then

$$\begin{aligned} & q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = & \ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ - & \ U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1) \\ & \qquad + a_1[P(\mathbf{Z}_1)+P(\mathbf{Z}_2)+P(\mathbf{Z}_3)], \\ F(\mathbf{Z}_1,\mathbf{Z}_2) = & \ G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)+(a_1-a_2)P(\mathbf{Z}_1), \end{aligned}$$

where $P:\ \mathcal{V}\mapsto\mathcal{V}$ and $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions, and

$$\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\
&=\frac{1}{a_1-a_2}\big\{a_1\big[-G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1)+G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)\big]\\
&\qquad -a_2[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)]\big\}+(a_1-a_2)P(\mathbf{Z}_2)\\
&+U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&-U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1).
\end{aligned}$$

Example 4.36. The general solution of the functional equation

$$\begin{aligned}
&2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-3f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
=\ &-3f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+2f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)
\end{aligned}$$

is given by the formula

$$\begin{aligned}
f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\ =\ &-2G(\mathbf{Z}_1,\mathbf{Z}_2)+2G(\mathbf{Z}_2,\mathbf{Z}_1)+2G(\mathbf{Z}_1,\mathbf{Z}_3)-2G(\mathbf{Z}_3,\mathbf{Z}_1)\\
&\quad-\ G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2)+P(\mathbf{Z}_2)\\
+\ &\ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
-\ &\ U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),
\end{aligned}$$

where $U:\ \mathcal{V}^3\mapsto\mathcal{V},\ G:\ \mathcal{V}^2\mapsto\mathcal{V}$ and $P:\ \mathcal{V}\mapsto\mathcal{V}$ are arbitrary functions.

Finally we consider the case when

$$(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2=0$$

and the complex constants $a_1,\ a_2,\ a_3$ are all distinct. This means that $a_1\omega+a_2\omega^2+a_3=0$ and equation (4.1) can be written as

(4.75)
$$\begin{aligned}
&a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)]-\omega^2a_2[f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)-\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)]\\
&\qquad=\alpha_1F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_3F(\mathbf{Z}_3,\mathbf{Z}_1),
\end{aligned}$$

where $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)=F(\mathbf{Z}_1,\mathbf{Z}_2)$. Also from (4.2) and (4.3) it follows that

(4.76)
$$\begin{aligned}
&a_1[f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-\omega f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)]-\omega^2a_2[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)]\\
&\qquad=\alpha_1F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_2F(\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_3F(\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}$$

(4.77)
$$\begin{aligned}
&a_1[f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)-\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)]-\omega^2a_2[f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-\omega f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)]\\
&\qquad=\alpha_1F(\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_2F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_3F(\mathbf{Z}_2,\mathbf{Z}_3).
\end{aligned}$$

From (4.75), (4.76) and (4.77) we obtain

$$(\alpha_1\omega+\alpha_2\omega^2+\alpha_3)\big[F(\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2F(\mathbf{Z}_2,\mathbf{Z}_3)+\omega F(\mathbf{Z}_3,\mathbf{Z}_1)\big]=\mathbf{O}.$$

First let us suppose that $\alpha_1\omega + \alpha_2\omega^2 + \alpha_3 \neq 0$. Then the following condition must be satisfied

$$F(\mathbf{Z}_1, \mathbf{Z}_2) + \omega^2 F(\mathbf{Z}_2, \mathbf{Z}_3) + \omega F(\mathbf{Z}_3, \mathbf{Z}_1) = \mathbf{O}.$$

This cyclic functional equation has the general solution

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = Q(\mathbf{Z}_1) - \omega^2 Q(\mathbf{Z}_2), \tag{4.78}$$

where Q is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

Let us suppose that $a_1^3 \neq a_2^3$. Then from (4.75), (4.76) and (4.77), if we take into account (4.78), we find that the function

$$\begin{aligned}
&g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \\
&= f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + \frac{\omega^2}{a_1^3 - a_2^3}\left\{a_1^2\big[\alpha_1 Q(\mathbf{Z}_2) + \alpha_2 Q(\mathbf{Z}_3) + \alpha_3 Q(\mathbf{Z}_1)\big]\right. \\
&+ \omega a_2^2\big[\alpha_1 Q(\mathbf{Z}_3) + \alpha_2 Q(\mathbf{Z}_1) + \alpha_3 Q(\mathbf{Z}_2)\big] \\
&\left.+ \omega^2 a_1 a_2\big[\alpha_1 Q(\mathbf{Z}_1) + \alpha_2 Q(\mathbf{Z}_2) + \alpha_3 Q(\mathbf{Z}_3)\big]\right\}
\end{aligned}$$

satisfies the cyclic equation

$$g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - \omega g(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = \mathbf{O}.$$

The last equation has the general solution

$$g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + \omega^2 p(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + \omega p(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2),$$

i.e.,

$$\begin{aligned}
&f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + \frac{\omega^2}{a_1^3 - a_2^3}\left\{a_1^2\big[\alpha_1 Q(\mathbf{Z}_2) + \alpha_2 Q(\mathbf{Z}_3) + \alpha_3 Q(\mathbf{Z}_1)\big]\right. \\
&+ \omega a_2^2\big[\alpha_1 Q(\mathbf{Z}_3) + \alpha_2 Q(\mathbf{Z}_1) + \alpha_3 Q(\mathbf{Z}_2)\big] \\
&\left.+ \omega^2 a_1 a_2\big[\alpha_1 Q(\mathbf{Z}_1) + \alpha_2 Q(\mathbf{Z}_2) + \alpha_3 Q(\mathbf{Z}_3)\big]\right\} \\
&= p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + \omega^2 p(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) + \omega p(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2),
\end{aligned} \tag{4.79}$$

where p is an arbitrary function $\mathcal{V}^3 \mapsto \mathcal{V}$.

By virtue of (4.78)

$$f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) = Q(\mathbf{Z}_1) - \omega^2 Q(\mathbf{Z}_2),$$

then from (4.79) it follows that

$$\begin{aligned}
&Q(\mathbf{Z}_1) - \omega^2 Q(\mathbf{Z}_2) + \frac{\omega^2}{a_1^3 - a_2^3}\left\{a_1^2\big[(\alpha_1 + \alpha_3) Q(\mathbf{Z}_1) + \alpha_2 Q(\mathbf{Z}_2)\big]\right. \\
&+ \omega a_2^2\big[(\alpha_2 + \alpha_3) Q(\mathbf{Z}_1) + \alpha_1 Q(\mathbf{Z}_2)\big] \\
&\left.+ \omega^2 a_1 a_2\big[(\alpha_1 + \alpha_2) Q(\mathbf{Z}_1) + \alpha_3 Q(\mathbf{Z}_2)\big]\right\} \\
&= p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1) + \omega p(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_1).
\end{aligned} \tag{4.80}$$

For $\mathbf{Z}_2 = \mathbf{Z}_1$ this equality takes the form

$$\left[1-\omega^2+\frac{\omega^2}{a_1-\omega^2 a_2}(\alpha_1+\alpha_2+\alpha_3)\right]Q(\mathbf{Z}_1)=\mathbf{O}$$

which implies that

$$\left[(\omega-1)(a_1-\omega^2 a_2)+\alpha_1+\alpha_2+\alpha_3\right]Q(\mathbf{Z}_1)=\mathbf{O}.$$

If

$$(\omega-1)(a_1-\omega^2 a_2)+\alpha_1+\alpha_2+\alpha_3\neq 0,$$

i.e., $a_1+a_2+a_3\neq\alpha_1+\alpha_2+\alpha_3$, then $Q(\mathbf{Z}_1)\equiv\mathbf{O}$ and (4.79) takes the form

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2), \tag{4.81}$$

where $p:\ \mathcal{V}^3\mapsto\mathcal{V}$ must satisfy the condition

$$p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)=\mathbf{O}. \tag{4.82}$$

Now we apply

Lemma 4.5. *Let $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)$ be given by the equality* (4.81)*, where $p:\ \mathcal{V}^3\mapsto\mathcal{V}$ satisfies* (4.82)*. Then*

$$p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega^2 U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1), \tag{4.83}$$

where $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function, and $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\equiv\mathbf{O}$.

Proof. We are looking for a function p in the form

$$\begin{aligned}p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= k_1 q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+k_2 q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+k_3 q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &+ k_4 q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)+k_5 q(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)+k_6 q(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),\end{aligned}$$

where k_i $(1\le i\le 6)$ are complex constants, satisfying (4.82) for an arbitrary function $q:\ \mathcal{V}^3\mapsto\mathcal{V}$. By a substitution into (4.82) we obtain

$$k_3=-\,\omega k_1-\omega^2 k_2,\qquad k_6=-\,\omega^2 k_4-\omega k_5.$$

By putting

$$\begin{aligned}U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= k_2 q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-\omega k_1 q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &+ k_4 q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-\omega k_5 q(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1)\end{aligned}$$

we obtain the representation (4.83). Now from (4.81) it is easy to see that $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\equiv\mathbf{O}$. □

Example 4.37. The general solution of the equation

$$\begin{aligned}& f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-i\sqrt{3}f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ =\;& \alpha_1 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_3 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1),\end{aligned}$$

where $\alpha_1+\alpha_2+\alpha_3\neq-\,i\sqrt{3}$ and $\alpha_1\omega+\alpha_2\omega^2+\alpha_3\neq 0$, is $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\equiv\mathbf{O}$.

Next we suppose that

$$(\omega - 1)(a_1 - \omega^2 a_2) + \alpha_1 + \alpha_2 + \alpha_3 = 0.$$

In this case the equality (4.78) can be written in the form

$$\gamma[Q(\mathbf{Z}_1) - Q(\mathbf{Z}_2)] = p(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1) + \omega p(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_1), \tag{4.84}$$

where

$$\gamma = \omega^2 \left(1 - \frac{a_1^2\alpha_2 + \omega a_2^2 \alpha_1 + \omega^2 a_1 a_2 \alpha_3}{a_1^3 - a_2^3}\right).$$

If $\gamma = 0$, we have as above

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1)$$

and (4.79) takes on the form

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = &- \frac{\omega^2}{a_1^3 - a_2^3} \left\{ a_1^2 \big[\alpha_1 Q(\mathbf{Z}_2) + \alpha_2 Q(\mathbf{Z}_3) + \alpha_3 Q(\mathbf{Z}_1)\big] \right. \\ &+ \omega a_2^2 \big[\alpha_1 Q(\mathbf{Z}_3) + \alpha_2 Q(\mathbf{Z}_1) + \alpha_3 Q(\mathbf{Z}_2)\big] \\ &\left. + \omega^2 a_1 a_2 \big[\alpha_1 Q(\mathbf{Z}_1) + \alpha_2 Q(\mathbf{Z}_2) + \alpha_3 Q(\mathbf{Z}_3)\big] \right\}. \end{aligned}$$

Example 4.38. The general solution of the functional equation

$$\begin{aligned} &f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) - \omega^2 f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) \\ = \; &-\omega^2 f(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2) + \beta f(\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_3) + (1-\beta) f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1), \end{aligned}$$

where β is an arbitrary constant, is

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = \beta Q(\mathbf{Z}_1) + (1-\beta) Q(\mathbf{Z}_2) - \omega^2 Q(\mathbf{Z}_3),$$

where Q is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$.

If $\gamma \neq 0$, then the general solution of (4.84) is

$$p(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = U(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) - \gamma Q(\mathbf{Z}_3)$$

and (4.80) takes on the form

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = &-\omega Q(\mathbf{Z}_1) - Q(\mathbf{Z}_2) - \omega^2 Q(\mathbf{Z}_3) \\ - \frac{\omega^2}{a_1^3 - a_2^3} &\left\{ \big[a_1^2(\alpha_3 - \alpha_2\omega^2) - a_2^2(\alpha_1 - \alpha_2\omega) - a_1 a_2 \omega(\alpha_3 - \alpha_1\omega)\big] Q(\mathbf{Z}_1) \right. \\ &\left. + \big[a_1^2(\alpha_1 - \alpha_2\omega) + a_2^2\omega(\alpha_3 - \alpha_1\omega) - a_1 a_2(\alpha_3 - \alpha_2\omega^2)\big] Q(\mathbf{Z}_2) \right\}. \end{aligned}$$

Example 4.39. The general solution of the functional equation

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - f(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_1) - i\sqrt{3} f(\mathbf{Z}_3, \mathbf{Z}_1, \mathbf{Z}_2) = -i\sqrt{3} f(\mathbf{Z}_3, \mathbf{Z}_3, \mathbf{Z}_1)$$

is given by the formula

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = (5 - i\sqrt{3}) Q(\mathbf{Z}_1) + (i\sqrt{3} - 1) Q(\mathbf{Z}_2) + 2(1 + i\sqrt{3}) Q(\mathbf{Z}_3),$$

where $Q: \; \mathcal{V} \mapsto \mathcal{V}$ is an arbitrary function.

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Now let us consider the case $a_1^3 = a_2^3$. This is possible if one of the following three equalities holds: $a_2 = a_1$, $a_2 = a_1\omega$ or $a_2 = a_1\omega^2$.

By assumption the coefficients a_1, a_2, a_3 are all distinct. We suppose that $a_2 = a_1\omega$, then $a_3 = a_1\omega^2$. Now the equations (4.1) and (4.2) can be written as

$$\begin{aligned}
&a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)]\\
&= \alpha_1 F(\mathbf{Z}_1,\mathbf{Z}_2) + \alpha_2 F(\mathbf{Z}_2,\mathbf{Z}_3) + \alpha_3 F(\mathbf{Z}_3,\mathbf{Z}_1),\\
&a_1[\omega^2 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)]\\
&= \alpha_1 F(\mathbf{Z}_2,\mathbf{Z}_3) + \alpha_2 F(\mathbf{Z}_3,\mathbf{Z}_1) + \alpha_3 F(\mathbf{Z}_1,\mathbf{Z}_2).
\end{aligned}$$

From these two equalities we derive

(4.85)
$$(\alpha_1 - \omega^2\alpha_3)F(\mathbf{Z}_1,\mathbf{Z}_2) + (\alpha_2 - \omega^2\alpha_1)F(\mathbf{Z}_2,\mathbf{Z}_3) + (\alpha_3 - \omega^2\alpha_2)F(\mathbf{Z}_3,\mathbf{Z}_1) = \mathbf{O}.$$

According to (4.78) we have

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = Q(\mathbf{Z}_1) - \omega^2 Q(\mathbf{Z}_2).$$

We substitute this into (4.84) and obtain $\alpha_2 = \alpha_1\omega^2$, $\alpha_3 = \alpha_1\omega$ (otherwise $Q(\mathbf{Z}_1) \equiv \mathbf{O}$). Thus (4.1) by virtue of (4.78) implies

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) = \mathbf{O}. \tag{4.86}$$

The general solution of this equation is

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \omega p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1). \tag{4.87}$$

Now we will use

Lemma 4.6. *Let the function* $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)$ *given by* (4.87) *satisfy* $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) = \mathbf{O}$. *Then*

$$\begin{aligned}
f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&\quad - U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3) - U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2) - U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),
\end{aligned} \tag{4.88}$$

where $U:\ \mathcal{V}^3 \mapsto \mathcal{V}$ *is an arbitrary function.*

Proof. We are looking for a function of the form

$$\begin{aligned}
p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= k_1 q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + k_2 q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + k_3 q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&+ k_4 q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3) + k_5 q(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2) + k_6 q(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1),
\end{aligned}$$

where k_i $(1 \le i \le 6)$ are complex constants, satisfying the equality

$$p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) - \omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) = \mathbf{O} \tag{4.89}$$

for an arbitrary function $q:\ \mathcal{V}^3 \mapsto \mathcal{V}$. By a substitution into (4.89) we find

$$k_3 = -\omega k_1 - \omega^2 k_2, \qquad k_5 = -k_1 + \omega^2(k_2 + k_4), \qquad k_6 = \omega(k_1 + k_4) - k_2$$

and

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= (k_2-\omega k_1)[q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+q(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+q(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ &- (k_2-\omega k_1)[q(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)+q(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)+q(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1)]. \end{aligned}$$

We obtain the representation (4.88) by putting

$$U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=(k_2-\omega k_1)q(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3). \qquad \square$$

The general solution of equation (4.86) satisfying the condition (4.78) is

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=Q(\mathbf{Z}_1)-\omega^2Q(\mathbf{Z}_3) \\ + \; & U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ - \; & U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1). \end{aligned} \tag{4.90}$$

Thus the general solution of the equation

$$\begin{aligned} & a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ & =\alpha_1[f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\omega f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)] \end{aligned}$$

is given by the formula (4.90), where $Q:\ \mathcal{V}\mapsto\mathcal{V}$ and $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions.

On the other hand, the general solution of the equation

$$\begin{aligned} & a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ & =\alpha_1 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_3 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1), \end{aligned}$$

is given by the formula (4.88) when $\alpha_1\omega+\alpha_2\omega^2+\alpha_3\neq 0$ and at least one of the equalities $\alpha_2=\alpha_1\omega^2$ and $\alpha_3=\alpha_1\omega$ is not satisfied.

Next we suppose that $a_2=a_1\omega^2$, then $a_3=-2a_1\omega$. Now the equations (4.1) and (4.2) can be written as

$$\begin{aligned} & a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ & =\alpha_1F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_3F(\mathbf{Z}_3,\mathbf{Z}_1), \\ & a_1[-2\omega f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ & =\alpha_3F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_1F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_2F(\mathbf{Z}_3,\mathbf{Z}_1). \end{aligned}$$

From these two equations, by virtue of (4.78), we obtain

$$\begin{aligned} & 3a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2\left[(\alpha_1-\omega^2\alpha_3)Q(\mathbf{Z}_2)\right. \\ & \left.+(\alpha_2-\omega^2\alpha_1)Q(\mathbf{Z}_3)+(\alpha_3-\omega^2\alpha_2)Q(\mathbf{Z}_1)\right] \\ = \; & \omega\left\{3a_1f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2\left[(\alpha_1-\omega^2\alpha_3)Q(\mathbf{Z}_1)\right.\right. \\ & \left.\left.+\ (\alpha_2-\omega^2\alpha_1)Q(\mathbf{Z}_2)+(\alpha_3-\omega^2\alpha_2)Q(\mathbf{Z}_3)\right]\right\}. \end{aligned}$$

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The general solution of this functional equation is

$$\begin{aligned} (4.91) \qquad & 3a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 \left[(\alpha_1 - \omega^2\alpha_3)Q(\mathbf{Z}_2)\right. \\ & \left. + (\alpha_2 - \omega^2\alpha_1)Q(\mathbf{Z}_3) + (\alpha_3 - \omega^2\alpha_2)Q(\mathbf{Z}_1)\right] \\ = \;& p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2), \end{aligned}$$

where p is an arbitrary function $\mathcal{V}^3 \mapsto \mathcal{V}$. It must satisfy the relation

$$\begin{aligned} (4.92) \qquad & 3a_1[Q(\mathbf{Z}_1) - \omega^2 Q(\mathbf{Z}_2)] + \omega^2 \left[(\alpha_1 - \omega^2\alpha_3)Q(\mathbf{Z}_1)\right. \\ & \left. + (\alpha_2 - \omega^2\alpha_1)Q(\mathbf{Z}_2) + (\alpha_3 - \omega^2\alpha_2)Q(\mathbf{Z}_1)\right] \\ = \;& p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + \omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1). \end{aligned}$$

For $\mathbf{Z}_2 = \mathbf{Z}_1$ this equality takes the form

$$(1-\omega^2)\left[3a_1 + \omega^2(\alpha_1+\alpha_2+\alpha_3)\right] Q(\mathbf{Z}_1) = \mathbf{O}.$$

If $3a_1 + \omega^2(\alpha_1+\alpha_2+\alpha_3) \neq 0$, then $Q(\mathbf{Z}_1) \equiv \mathbf{O}$ and (4.91) takes the form

$$3a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),$$

where $p:\ \mathcal{V}^3 \mapsto \mathcal{V}$ must satisfy the condition (4.82). Thus again $p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)$ is given by (4.83) and $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}$. So the general solution of the equation

$$\begin{aligned} & a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - 2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ = \;& \alpha_1 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \alpha_2 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) + \alpha_3 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1), \end{aligned}$$

where $\alpha_1+\alpha_2+\alpha_3 \neq -\,3a_1\omega$, is $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}$.

Now suppose that $3a_1\omega + \alpha_1 + \alpha_2 + \alpha_3 = 0$. In this case the equality (4.91) can be written in the form (4.84), where

$$\gamma = \omega(3a_1\omega + \alpha_1 - \alpha_2\omega).$$

If $\gamma = 0$, we have as above

$$p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)$$

and (4.91) takes the form

$$(4.93) \qquad f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = Q(\mathbf{Z}_2) - \omega^2 Q(\mathbf{Z}_3).$$

Thus the general solution of the equation

$$\begin{aligned} & a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - 2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)] \\ = \;& (\beta - 3a_1)\omega f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \beta f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2\beta f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1), \end{aligned}$$

where β is a complex constant, is given by the formula (4.93), where $Q:\ \mathcal{V} \mapsto \mathcal{V}$ is an arbitrary function.

If $\gamma \neq 0$, then as above

$$p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - \gamma Q(\mathbf{Z}_3)$$

and (4.91) takes on the form

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= -\left[\omega Q(\mathbf{Z}_1)+Q(\mathbf{Z}_2)+\omega^2 Q(\mathbf{Z}_3)\right] \\ + & \frac{1}{3a_1}\left[\big(-\alpha_1+2\alpha_2\omega-\alpha_3\omega^2\big)Q(\mathbf{Z}_1)+\big(-2\alpha_1\omega^2+\alpha_2+\alpha_3\omega\big)Q(\mathbf{Z}_2)\right].\end{aligned}$$

Example 4.40. The general solution of the functional equation

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = & \ \omega f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-(1+4\omega)f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=(1-\omega)Q(\mathbf{Z}_1)+\omega Q(\mathbf{Z}_2)-\omega^2 Q(\mathbf{Z}_3),$$

where $Q:\ \mathcal{V}\mapsto\mathcal{V}$ is an arbitrary function.

Now we pass on to the case $\alpha_1\omega+\alpha_2\omega^2+\alpha_3=0$. We suppose that $a_1^3\neq a_2^3$. Then from (4.75), (4.76) and (4.77) we obtain that the function

$$\begin{aligned} & g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \\ & = f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\frac{1}{a_1^3-a_2^3}\left\{a_1^2[\alpha_1F(\mathbf{Z}_1,\mathbf{Z}_2)-\alpha_2\omega^2F(\mathbf{Z}_3,\mathbf{Z}_1)]\right. \\ & +\omega a_2^2[\alpha_1F(\mathbf{Z}_2,\mathbf{Z}_3)-\alpha_2\omega^2F(\mathbf{Z}_1,\mathbf{Z}_2)] \\ & \left.+\omega^2a_1a_2[\alpha_1F(\mathbf{Z}_3,\mathbf{Z}_1)-\alpha_2\omega^2F(\mathbf{Z}_2,\mathbf{Z}_3)]\right\}\end{aligned}$$

satisfies the cyclic equation

$$g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega g(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)=\mathbf{O}.$$

Since the general solution of this equation is

$$g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),$$

we obtain

$$\begin{aligned} \text{(4.94)}\qquad f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{1}{a_1^3-a_2^3}\left[(a_1^2\alpha_1-a_2^2\alpha_2)F(\mathbf{Z}_1,\mathbf{Z}_2)\right. \\ + & \left.\omega a_2(a_2\alpha_1-a_1\alpha_2)F(\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2a_1(a_2\alpha_1-a_1\alpha_2)F(\mathbf{Z}_3,\mathbf{Z}_1)\right] \\ + & \ p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

where $p:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function.

From the equality (4.94) we obtain

$$\begin{aligned} \text{(4.95)}\qquad F(\mathbf{Z}_1,\mathbf{Z}_2) &= \frac{1}{a_1^3-a_2^3}\left[(a_1^2\alpha_1-a_2^2\alpha_2)F(\mathbf{Z}_1,\mathbf{Z}_1)\right. \\ + & \left.\omega a_2(a_2\alpha_1-a_1\alpha_2)F(\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2a_1(a_2\alpha_1-a_1\alpha_2)F(\mathbf{Z}_2,\mathbf{Z}_1)\right] \\ + & \ p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned}$$

For $\mathbf{Z}_2 = \mathbf{Z}_1$ (4.95) yields

$$F(\mathbf{Z}_1,\mathbf{Z}_1) = \frac{\alpha_1 - \alpha_2\omega^2}{a_1 - a_2\omega^2} F(\mathbf{Z}_1,\mathbf{Z}_1). \tag{4.96}$$

First we consider the case

$$\alpha_1 - \alpha_2\omega^2 = a_1 - a_2\omega^2,$$

then $F(\mathbf{Z}_1,\mathbf{Z}_1) = Q(\mathbf{Z}_1)$, where Q is an arbitrary function $\mathcal{V} \mapsto \mathcal{V}$. Now we have

$$\begin{aligned}&\left[1 + \frac{\omega a_2(\alpha_1 - a_1)}{a_1^2\omega^2 + a_1a_2\omega + a_2^2}\right] F(\mathbf{Z}_1,\mathbf{Z}_2) + \frac{\omega^2 a_1(\alpha_1 - a_1)}{a_1^2\omega^2 + a_1a_2\omega + a_2^2} F(\mathbf{Z}_2,\mathbf{Z}_1)\\ &= \frac{\alpha_1\omega(a_1\omega + a_2) + a_2^2}{a_1^2\omega^2 + a_1a_2\omega + a_2^2} Q(\mathbf{Z}_1) + p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)\\ &+ \omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + \omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned} \tag{4.97}$$

First we consider the particular case $\alpha_1 = a_1$ (then $\alpha_2 = a_2,\ \alpha_3 = a_3 = -(a_1\omega + a_2\omega^2)$). Now

$$F(\mathbf{Z}_1,\mathbf{Z}_2) = Q(\mathbf{Z}_1) + p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + \omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).$$

Thus we find that the functional equation

$$\begin{aligned}&a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - (a_1\omega + a_2\omega^2) f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ =\ &a_1 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + a_2 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) - (a_1\omega + a_2\omega^2) f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

has the general solution

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= Q(\mathbf{Z}_1) + p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + \omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)\\ &+ p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

where $Q:\ \mathcal{V} \mapsto \mathcal{V}$ and $p:\ \mathcal{V}^3 \mapsto \mathcal{V}$ are arbitrary functions.

Now we consider equation (4.97) in the general case. By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we derive the equation

$$\begin{aligned}&\frac{\omega^2 a_1(\alpha_1 - a_1)}{a_1^2\omega^2 + a_1a_2\omega + a_2^2} F(\mathbf{Z}_1,\mathbf{Z}_2) + \left[1 + \frac{\omega a_2(\alpha_1 - a_1)}{a_1^2\omega^2 + a_1a_2\omega + a_2^2}\right] F(\mathbf{Z}_2,\mathbf{Z}_1)\\ &= \frac{\alpha_1\omega(a_1\omega + a_2) + a_2^2}{a_1^2\omega^2 + a_1a_2\omega + a_2^2} Q(\mathbf{Z}_2) + p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)\\ &+ \omega^2 p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2) + \omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2).\end{aligned} \tag{4.98}$$

The determinant of the system (4.97), (4.98) is

$$\frac{[a_2^2 + \alpha_1\omega(a_2 + a_1\omega)][2a_1^2\omega^2 + a_2^2 + \alpha_1\omega(a_2 - a_1\omega)]}{(a_1^2\omega^2 + a_1a_2\omega + a_2^2)^2}. \tag{4.99}$$

If this expression is not 0, then the solution of this system is

$$
\begin{aligned}
F(\mathbf{Z}_1,\mathbf{Z}_2) &= \frac{(a_1^2\omega^2+a_2^2+a_2\alpha_1\omega)Q(\mathbf{Z}_1)-\omega^2 a_1(\alpha_1-a_1)Q(\mathbf{Z}_2)}{2a_1^2\omega^2+a_2^2+\alpha_1\omega(a_2-a_1\omega)} \\
&+ \frac{a_1^2\omega^2+a_1a_2\omega+a_2^2}{[a_2^2+\alpha_1\omega(a_2+a_1\omega)][2a_1^2\omega^2+a_2^2+\alpha_1\omega(a_2-a_1\omega)]} \\
&\times \left\{(a_1^2\omega^2+a_2^2+a_2\alpha_1\omega)[p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2 p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)]\right. \\
&\left. -\ \omega^2 a_1(\alpha_1-a_1)[p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+\omega^2 p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)]\right\}.
\end{aligned}
$$

Example 4.41. The general solution of the equation

$$
\begin{aligned}
&2f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-2i\sqrt{3}f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\
&=(1-i\sqrt{3})f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-(1+i\sqrt{3})f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)
\end{aligned}
$$

is given by the formula

$$
\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \\
&=\frac{1}{4}\left[(1-i\sqrt{3})F(\mathbf{Z}_1,\mathbf{Z}_2)+(1+i\sqrt{3})F(\mathbf{Z}_2,\mathbf{Z}_3)+2F(\mathbf{Z}_3,\mathbf{Z}_1)\right] \\
&+\ p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}
$$

where

$$
\begin{aligned}
F(\mathbf{Z}_1,\mathbf{Z}_2) &= (3-i\sqrt{3})Q(\mathbf{Z}_1)+2Q(\mathbf{Z}_2) \\
+ &\quad (3+i\sqrt{3})[p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)] \\
+ &\quad (1+i\sqrt{3})[p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+\omega^2 p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)],
\end{aligned}
$$

$Q:\ \mathcal{V}\mapsto\mathcal{V}$ and $p:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions and $\omega=\frac{-1+i\sqrt{3}}{2}$.

Now let us suppose that the expression (4.99) is 0. First let

$$a_2^2+\alpha_1\omega(a_2+a_1\omega)=0.$$

If $a_2+a_1\omega=0$, then $a_1=a_2=a_3=0$ which is a contradiction. Thus

$$\alpha_1=-\frac{a_2^2\omega^2}{a_2+a_1\omega}, \qquad \alpha_2=-\frac{a_1^2\omega^2}{a_2+a_1\omega}.$$

Now (4.94) takes the form

$$
\begin{aligned}
f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{a_2F(\mathbf{Z}_2,\mathbf{Z}_3)+a_1\omega F(\mathbf{Z}_3,\mathbf{Z}_1)}{a_2+a_1\omega} \\
+ &\quad p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}
$$

while (4.97) becomes

$$\frac{a_1\omega}{a_2+a_1\omega}[F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_1)]=p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)\\ +\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).$$

The last equation implies

$$\begin{aligned} p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega^2U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-a_1P(\mathbf{Z}_1),\\ F(\mathbf{Z}_1,\mathbf{Z}_2) &= G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1)+(a_2+a_1\omega)P(\mathbf{Z}_1),\end{aligned}$$

where $P:\ \mathcal{V}\mapsto\mathcal{V},\ G:\mathcal{V}^2\mapsto\mathcal{V}$ and $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions.

Thus the general solution of the functional equation

$$\begin{aligned} &a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-(a_1\omega+a_2\omega^2)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ &=-\frac{a_2^2\omega^2}{a_2+a_1\omega}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-\frac{a_1^2\omega^2}{a_2+a_1\omega}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)\\ &+\frac{a_2^2+a_1^2\omega}{a_2+a_1\omega}f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned} g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= \frac{a_2[G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2)]+a_1\omega[G(\mathbf{Z}_3,\mathbf{Z}_1)+G(\mathbf{Z}_1,\mathbf{Z}_3)]}{a_2+a_1\omega}\\ &\quad -a_1P(\mathbf{Z}_1)+(a_2-a_1\omega^2)P(\mathbf{Z}_2).\end{aligned}$$

Next we suppose that

$$2a_1^2\omega^2+a_2^2-\alpha_1\omega(a_1\omega-a_2)=0.$$

Since $a_1^3\neq a_2^3$, we have

$$\alpha_1=\frac{2a_1^2\omega+a_2^2\omega^2}{a_1\omega-a_2},\qquad \alpha_2=\frac{a_1^2\omega^2+2a_1a_2\omega}{a_1\omega-a_2}.$$

Now (4.94) takes the form

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\frac{1}{a_1\omega-a_2}\left[2a_1\omega F(\mathbf{Z}_1,\mathbf{Z}_2)-a_1\omega F(\mathbf{Z}_3,\mathbf{Z}_1)-a_2F(\mathbf{Z}_2,\mathbf{Z}_3)\right]\\ +p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),$$

while (4.97) becomes

$$\begin{aligned} &\frac{a_1\omega}{a_1\omega-a_2}\left[F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_2,\mathbf{Z}_1)-2F(\mathbf{Z}_1,\mathbf{Z}_1)\right]\\ =\ &p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).\end{aligned}$$

The last equation implies

$$\begin{aligned} p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega^2U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_1P(\mathbf{Z}_1),\\ F(\mathbf{Z}_1,\mathbf{Z}_2) &= G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)+(a_1\omega-a_2)P(\mathbf{Z}_1),\end{aligned}$$

where $P:\ \mathcal{V}\mapsto\mathcal{V},\ G:\mathcal{V}^2\mapsto\mathcal{V}$ and $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions.

Thus the general solution of the functional equation

$$\begin{aligned}
&a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-(a_1\omega+a_2\omega^2)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&=\frac{2a_1^2\omega+a_2^2\omega^2}{a_1\omega-a_2}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)\\
&\quad+\frac{a_1^2\omega^2+2a_1a_2\omega}{a_1\omega-a_2}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)\\
&\quad-\frac{a_1^2(\omega^2-1)+2a_1a_2+a_2^2}{a_1\omega-a_2}f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)
\end{aligned}$$

is given by the formula

$$\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\
&=\frac{1}{a_1\omega-a_2}\big\{2a_1\omega\big[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)\big]\\
&+a_1\omega\big[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)\big]-a_2\big[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)\big]\big\}\\
&\qquad+a_1(1+2\omega)P(\mathbf{Z}_1)+(a_1\omega^2-a_2)P(\mathbf{Z}_2).
\end{aligned}$$

Now let $\frac{\alpha_1-\alpha_2\omega^2}{a_1-a_2\omega^2}=\gamma\neq 1$. Then from (4.96) we find $F(\mathbf{Z}_1,\mathbf{Z}_1)\equiv\mathbf{O}$. Now we have

(4.100)
$$\begin{aligned}
&\left[1+\frac{\omega a_2(\alpha_1-a_1\gamma)}{a_1^2\omega^2+a_1a_2\omega+a_2^2}\right]F(\mathbf{Z}_1,\mathbf{Z}_2)+\frac{\omega^2a_1(\alpha_1-a_1\gamma)}{a_1^2\omega^2+a_1a_2\omega+a_2^2}F(\mathbf{Z}_2,\mathbf{Z}_1)\\
&\quad=p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).
\end{aligned}$$

By a permutation of the variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we derive the equation

(4.101)
$$\begin{aligned}
&\frac{\omega^2a_1(\alpha_1-a_1\gamma)}{a_1^2\omega^2+a_1a_2\omega+a_2^2}F(\mathbf{Z}_1,\mathbf{Z}_2)+\left[1+\frac{\omega a_2(\alpha_1-a_1\gamma)}{a_1^2\omega^2+a_1a_2\omega+a_2^2}\right]F(\mathbf{Z}_2,\mathbf{Z}_1)\\
&\quad=p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2).
\end{aligned}$$

The determinant of the system (4.100), (4.101) is

(4.102)
$$\begin{aligned}
&\frac{(1-\gamma)(a_1^2\omega^2+a_1a_2\omega)+a_2^2+\omega\alpha_1(a_1\omega+a_2)}{\big(a_1^2\omega^2+a_1a_2\omega+a_2^2\big)^2}\\
&\qquad\times\big[(1+\gamma)a_1^2\omega^2+(1-\gamma)a_1a_2\omega+a_2^2-\omega\alpha_1(a_1\omega-a_2)\big].
\end{aligned}$$

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If this expression is not 0, then the solution of this system is

$$F(\mathbf{Z}_1,\mathbf{Z}_2)=\frac{a_1^2\omega^2+a_1a_2\omega+a_2^2}{(1-\gamma)(a_1^2\omega^2+a_1a_2\omega)+a_2^2+\omega\alpha_1(a_1\omega+a_2)}$$
$$\times\left[(1+\gamma)a_1^2\omega^2+(1-\gamma)a_1a_2\omega+a_2^2-\omega\alpha_1(a_1\omega-a_2)\right]^{-1}$$
$$\times\left\{\left[a_1^2\omega^2+(1-\gamma)a_1a_2\omega+a_2^2+a_2\alpha_1\omega\right]\right.$$
$$\times[p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)]$$
$$\left.-\omega^2a_1(\alpha_1-a_1\gamma)[p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)]\right\}.$$

Example 4.42. The general solution of the equation

$$\begin{aligned}&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-i\sqrt{3}f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\=\;&(1-i\sqrt{3})f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-(1+i\sqrt{3})f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$\begin{aligned}&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\&=\frac{1}{2}\left[(1-i\sqrt{3})F(\mathbf{Z}_1,\mathbf{Z}_2)+(1+i\sqrt{3})F(\mathbf{Z}_2,\mathbf{Z}_3)+2F(\mathbf{Z}_3,\mathbf{Z}_1)\right]\\&+p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

where

$$\begin{aligned}F(\mathbf{Z}_1,\mathbf{Z}_2)&=\frac{i\sqrt{3}}{3}\,[p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)]\\&+\frac{i\sqrt{3}-3}{6}[p(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\omega p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)],\end{aligned}$$

$p:\ \mathcal{V}^3\mapsto\mathcal{V}$ is an arbitrary function and $\omega=\frac{-1+i\sqrt{3}}{2}$.

Now let us suppose that the expression (4.102) is 0. First let

$$(1-\gamma)(a_1^2\omega^2+a_1a_2\omega)+a_2^2+\omega\alpha_1(a_1\omega+a_2)=0.$$

Then

$$\alpha_1=-\frac{a_2^2\omega^2+(1-\gamma)(a_1^2\omega+a_1a_2)}{a_1\omega+a_2},\quad \alpha_2=-\frac{a_1^2\omega^2+(1-\gamma)(a_2^2+a_1a_2\omega)}{a_1\omega+a_2}.$$

Now (4.94) takes the form

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=(\gamma-1)F(\mathbf{Z}_1,\mathbf{Z}_2)+\frac{a_1\omega F(\mathbf{Z}_3,\mathbf{Z}_1)+a_2F(\mathbf{Z}_2,\mathbf{Z}_3)}{a_1\omega+a_2}\\&+p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),\end{aligned}$$

while (4.100) becomes

$$\frac{a_1\omega}{a_1\omega + a_2}\big[F(\mathbf{Z}_1,\mathbf{Z}_2) - F(\mathbf{Z}_2,\mathbf{Z}_1)\big] = p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + \omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).$$

As in the case $\gamma = 1$, the last equation implies

$$\begin{aligned} p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - a_1 P(\mathbf{Z}_1), \\ F(\mathbf{Z}_1,\mathbf{Z}_2) &= G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1) + (a_1\omega + a_2)P(\mathbf{Z}_1), \end{aligned}$$

but the functions $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ and $P:\ \mathcal{V} \mapsto \mathcal{V}$ satisfy the relation

$$2G(\mathbf{Z}_1,\mathbf{Z}_1) + (a_1\omega + a_2)P(\mathbf{Z}_1) = \mathbf{O}.$$

Thus the general solution of the equation

$$\begin{aligned} & a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - (a_1\omega + a_2\omega^2) f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ &= -\frac{(1-\gamma)(a_1^2\omega + a_1a_2) + a_2^2\omega^2}{a_1\omega + a_2} f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) \\ &\quad - \frac{a_1^2\omega^2 + (1-\gamma)(a_2^2 + a_1a_2\omega)}{a_1\omega + a_2} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) \\ &\quad + \left[(1-\gamma)(a_1\omega + a_2\omega^2) + \frac{a_2^2 + a_1^2\omega^2}{a_1\omega + a_2}\right] f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is given by the formula

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = (\gamma - 1)\big[G(\mathbf{Z}_1,\mathbf{Z}_2) + G(\mathbf{Z}_2,\mathbf{Z}_1)\big] \\ + \quad & \frac{1}{a_1\omega + a_2}\big\{a_1\omega\big[G(\mathbf{Z}_1,\mathbf{Z}_3) + G(\mathbf{Z}_3,\mathbf{Z}_1)\big] + a_2\big[G(\mathbf{Z}_2,\mathbf{Z}_3) + G(\mathbf{Z}_3,\mathbf{Z}_2)\big] \\ & + 2\big[a_1 - (\gamma - 1)(a_1\omega + a_2)\big]G(\mathbf{Z}_1,\mathbf{Z}_1) + 2(a_1\omega^2 - a_2)G(\mathbf{Z}_2,\mathbf{Z}_2)\big\}. \end{aligned}$$

Next we suppose that

$$(1+\gamma)a_1^2\omega^2 + (1-\gamma)a_1a_2\omega + a_2^2 - \omega\alpha_1(a_1\omega - a_2) = 0.$$

In this case we have

$$\begin{aligned} \alpha_1 &= \frac{a_2^2\omega^2 + (1+\gamma)a_1^2\omega + (1-\gamma)a_1a_2}{a_1\omega - a_2}, \\ \alpha_2 &= \frac{a_1^2\omega^2 + (1-\gamma)a_2^2 + (1+\gamma)a_1a_2\omega}{a_1\omega - a_2}. \end{aligned}$$

Now (4.94) takes the form

$$\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\
&=\frac{1}{a_1\omega-a_2}\big\{\big[(1+\gamma)a_1\omega+(1-\gamma)a_2\big]F(\mathbf{Z}_1,\mathbf{Z}_2)-a_1\omega F(\mathbf{Z}_3,\mathbf{Z}_1)\\
&-a_2F(\mathbf{Z}_2,\mathbf{Z}_3)\big\}+p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}$$

while (4.100) becomes

$$\begin{aligned}
&\frac{a_1\omega}{a_1\omega-a_2}\big[F(\mathbf{Z}_1,\mathbf{Z}_2)+F(\mathbf{Z}_2,\mathbf{Z}_1)\big]\\
&=p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1).
\end{aligned}$$

As above we obtain

$$\begin{aligned}
p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega^2U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1),\\
F(\mathbf{Z}_1,\mathbf{Z}_2)&=G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1).
\end{aligned}$$

Thus the general solution of the equation

$$\begin{aligned}
&a_1f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-(a_1\omega+a_2\omega^2)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\
&=\frac{(1+\gamma)a_1^2\omega+(1-\gamma)a_1a_2+a_2^2\omega^2}{a_1\omega-a_2}f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)\\
&+\frac{a_1^2\omega^2+(1+\gamma)a_1a_2\omega+(1-\gamma)a_2^2}{a_1\omega-a_2}f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)\\
&+\frac{a_1^2(1-\gamma\omega^2)+a_1a_2[\omega^2+\gamma(\omega-1)]+a_2^2(\omega+\gamma\omega^2)}{a_1\omega-a_2}f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)
\end{aligned}$$

is given by the formula

$$\begin{aligned}
&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\
&=\frac{1}{a_1\omega-a_2}\big\{\big[(1+\gamma)a_1\omega+(1-\gamma)a_2\big]\big[G(\mathbf{Z}_1,\mathbf{Z}_2)-G(\mathbf{Z}_2,\mathbf{Z}_1)\big]\\
&+a_1\omega\,\big[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)\big]-a_2\big[G(\mathbf{Z}_2,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_2)\big]\big\},
\end{aligned}$$

where $G:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function.

Now we suppose that $a_1^3=a_2^3$. As above, this is possible if $a_2=a_1\omega$ or $a_2=a_1\omega^2$.

First let us suppose that $a_2=a_1\omega$, then $a_3=a_1\omega^2$. The equality (4.84) still holds. By virtue of $\alpha_1\omega+\alpha_2\omega^2+\alpha_3=0$ it takes the form

(4.103)
$$(2\alpha_1+\omega\alpha_2)F(\mathbf{Z}_1,\mathbf{Z}_2)+(\alpha_2-\omega^2\alpha_1)F(\mathbf{Z}_2,\mathbf{Z}_3)-(\omega\alpha_1+2\omega^2\alpha_2)F(\mathbf{Z}_3,\mathbf{Z}_1)=\mathbf{O}.$$

If $\alpha_2=\alpha_1\omega$ (and $\alpha_3=\alpha_1\omega^2$), this equation has the nontrivial solution

(4.104) $$F(\mathbf{Z}_1,\mathbf{Z}_2)=P(\mathbf{Z}_1)-\omega P(\mathbf{Z}_2),$$

where $P:\ \mathcal{V}\mapsto\mathcal{V}$ is an arbitrary function. Then equation (4.1) implies

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)=\mathbf{O}.$$

According to Lemma 4.6, the general solution of this equation satisfying the condition (4.104) is

$$\begin{aligned}f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=P(\mathbf{Z}_2)-\omega P(\mathbf{Z}_3)\\ +&\ U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+U(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ -&\ U(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_3)-U(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_2)-U(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_1).\end{aligned}\tag{4.105}$$

Thus the general solution of the equation

$$\begin{aligned}&a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)]\\ =\ &\alpha_1[f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)]\end{aligned}$$

is given by the formula (4.105), where $P:\ \mathcal{V}\mapsto\mathcal{V}$ and $U:\ \mathcal{V}^3\mapsto\mathcal{V}$ are arbitrary functions.

On the other hand, if $\alpha_2\neq\alpha_1\omega$, then the general solution of equation (4.103) is
$F(\mathbf{Z}_1,\mathbf{Z}_2)\equiv\mathbf{O}$. The general solution of the equation (4.86) satisfying $f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)\equiv\mathbf{O}$ is given by (4.88).

Thus the general solution of the equation

$$\begin{aligned}&a_1[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)]\\ =\ &\alpha_1 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2 f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-(\alpha_1\omega+\alpha_2\omega^2)f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1),\end{aligned}$$

where $\alpha_2\neq\alpha_1\omega$, is given by (4.88).

Next we suppose that $a_2=a_1\omega^2$, then $a_3=-2a_1\omega$. Now the equations (4.1) and (4.2) can be written as

$$\begin{aligned}&a_1\big[f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\big]\\ =\ &\alpha_1 F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_2 F(\mathbf{Z}_2,\mathbf{Z}_3)-(\alpha_1\omega+\alpha_2\omega^2)F(\mathbf{Z}_3,\mathbf{Z}_1),\\ &a_1\big[-2\omega f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\big]\\ =\ &-(\alpha_1\omega+\alpha_2\omega^2)F(\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_1 F(\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_2 F(\mathbf{Z}_3,\mathbf{Z}_1).\end{aligned}$$

From these two equations we obtain

(4.106)
$$\begin{aligned}&3a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+(2\alpha_1\omega^2+\alpha_2)F(\mathbf{Z}_2,\mathbf{Z}_3)+(\alpha_1\omega+2\alpha_2\omega^2)F(\mathbf{Z}_3,\mathbf{Z}_1)\\ &=\omega\left[3a_1 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)+(2\alpha_1\omega^2+\alpha_2)F(\mathbf{Z}_1,\mathbf{Z}_2)\right.\\ &\left.+\ (\alpha_1\omega+2\alpha_2\omega^2)F(\mathbf{Z}_2,\mathbf{Z}_3)\right].\end{aligned}$$

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The general solution of this functional equation is

(4.107)

$$3a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + (2\alpha_1\omega^2+\alpha_2)F(\mathbf{Z}_2,\mathbf{Z}_3) + (\alpha_1\omega+2\alpha_2\omega^2)F(\mathbf{Z}_3,\mathbf{Z}_1)$$
$$= p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 p(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + \omega p(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2),$$

where $p:\ \mathcal{V}^3 \mapsto \mathcal{V}$ is an arbitrary function. It must satisfy the relation

$$(3a_1+2\alpha_1\omega^2+\alpha_2)F(\mathbf{Z}_1,\mathbf{Z}_2) + (\alpha_1\omega+2\alpha_2\omega^2)F(\mathbf{Z}_2,\mathbf{Z}_1)$$
$$= p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + \omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1) + \omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1). \tag{4.108}$$

For $\mathbf{Z}_2 = \mathbf{Z}_1$ this equality takes the form

$$\big[3a_1 + (2\omega^2+\omega)\alpha_1 + (1+2\omega^2)\alpha_2\big]\, F(\mathbf{Z}_1,\mathbf{Z}_1) = \mathbf{O}.$$

If

$$3a_1 + (2\omega^2+\omega)\alpha_1 + (1+2\omega^2)\alpha_2 \neq 0,$$

then $F(\mathbf{Z}_1,\mathbf{Z}_1) \equiv \mathbf{O}$. Further on, if

$$3a_1 + 2\alpha_1\omega^2 + \alpha_2 \neq \alpha_1\omega + 2\alpha_2\omega^2,$$

from equation (4.108) we find

$$F(\mathbf{Z}_1,\mathbf{Z}_2) \equiv \mathbf{O}, \quad p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)$$

and $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}$.

Example 4.43. The general solution of the equation

$$\begin{aligned} & f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + \omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) - 2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = \; & \omega f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2) + f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3) - 2\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

is $f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) \equiv \mathbf{O}$.

If

$$3a_1 + 2\alpha_1\omega^2 + \alpha_2 = \alpha_1\omega + 2\alpha_2\omega^2,$$

then from (4.108) we find

$$\begin{aligned} F(\mathbf{Z}_1,\mathbf{Z}_2) &= G(\mathbf{Z}_1,\mathbf{Z}_2) - G(\mathbf{Z}_2,\mathbf{Z}_1), \\ p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - \omega^2 U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) \end{aligned}$$

and

$$\begin{aligned} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) &= (\alpha_1\omega + 2\alpha_2\omega^2)\big[G(\mathbf{Z}_1,\mathbf{Z}_3) - G(\mathbf{Z}_3,\mathbf{Z}_1)\big] \\ &+ (3a_1 - \alpha_1\omega - 2\alpha_2\omega^2)\big[G(\mathbf{Z}_2,\mathbf{Z}_3) - G(\mathbf{Z}_3,\mathbf{Z}_2)\big], \end{aligned}$$

where $G:\ \mathcal{V}^2 \mapsto \mathcal{V}$ is an arbitrary function.

Example 4.44. The general solution of the equation

$$(\omega^2-1)f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+(\omega-\omega^2)f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+2(\omega-1)f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)$$
$$=\omega f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)-2\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)$$

is given by the formula

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\omega^2\big[G(\mathbf{Z}_1,\mathbf{Z}_3)-G(\mathbf{Z}_3,\mathbf{Z}_1)\big]-G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2).$$

Now we suppose that

$$3a_1+(2\omega^2+\omega)\alpha_1+(1+2\omega^2)\alpha_2=0.$$

In this case the equality (4.108) can be written in the form

$$\gamma\big[F(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_2,\mathbf{Z}_1)\big]=p(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\omega^2 p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\omega p(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1),$$

where $\gamma=-\alpha_1\omega-2\alpha_2\omega^2$.

If $\gamma=0$, *i.e.*, $\alpha_1=-2a_1\omega,\ \alpha_2=a_1$, we have as above

$$p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega^2U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1).$$

Thus the general solution of the functional equation

$$\begin{aligned}&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\=\ &-2\omega f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=F(\mathbf{Z}_2,\mathbf{Z}_3),$$

where $F:\ \mathcal{V}^2\mapsto\mathcal{V}$ is an arbitrary function.

If $\gamma\neq0$, then

$$\begin{aligned}F(\mathbf{Z}_1,\mathbf{Z}_2)&=G(\mathbf{Z}_1,\mathbf{Z}_2)+G(\mathbf{Z}_2,\mathbf{Z}_1)-\omega P(\mathbf{Z}_1),\\p(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)&=U(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)-\omega^2U(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+\gamma P(\mathbf{Z}_1)\end{aligned}$$

and

$$\begin{aligned}&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\\&=G(\mathbf{Z}_1,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_1)+\left(\frac{3a_1}{\gamma}-1\right)\big[G(\mathbf{Z}_2,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_2)\big]\\&\quad+\ P(\mathbf{Z}_1)-\left(1+\frac{3a_1}{\gamma}\omega\right)P(\mathbf{Z}_2),\end{aligned}$$

where $G:\ \mathcal{V}^2\mapsto\mathcal{V}$ and $P:\ \mathcal{V}\mapsto\mathcal{V}$ are arbitrary functions.

Example 4.45. The general solution of the equation

$$\begin{aligned}&f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+\omega^2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)-2\omega f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\=\ &\omega^2 f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)-2\omega f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\end{aligned}$$

is given by the formula

$$f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=G(\mathbf{Z}_1,\mathbf{Z}_3)+G(\mathbf{Z}_3,\mathbf{Z}_1)+P(\mathbf{Z}_1)+\omega^2P(\mathbf{Z}_2).$$

5. Stability criteria

In this section we will derive a necessary and sufficient condition for the stability of the quasicyclic complex vector functional equation (3.1), *i.e.*, its matrix form (3.4) using a simple spectral property of compound matrices.

Let $\det A\neq 0$, then relation (3.4) takes the form

$$F=A^{-1}\Lambda G\equiv SG, \tag{5.1}$$

where S is also a cyclic complex matrix.

Definition 5.1. The quasicyclic complex vector functional equation (5.1) is *stable* if $\operatorname{stab}(S)<0$.

Proposition 5.2. *For any cyclic matrix $S\in\mathcal{V}$ it holds*

$$\operatorname{stab}(S)=\inf\{\mu(S),\ \mu\ \text{ is a Lozinskiĭ measure on } \mathcal{V}^n\}. \tag{5.2}$$

Proof. The relation (5.2) obviously holds for diagonizable matrices in view of

$$\mu_T(S)=\mu(TST^{-1})\qquad (T\ \text{ is an invertible matrix}) \tag{5.3}$$

and the first two relations in (2.4). Furthermore, the infimum in (5.2) can be achieved if S is diagonizable. The general case can be shown based on this observation, the fact that S can be approximated by diagonizable matrices in $\mathcal{V}$ and the continuity of $\mu(\cdot)$, which is implied by the property

$$|\mu(A)-\mu(B)|\leq|A-B|. \qquad \square$$

Remark 5.3. From the above proof it follows that

$$\operatorname{stab}(S)=\inf\{\mu_\infty(TST^{-1}),\ T\ \text{ is invertible}\}.$$

The same relation holds if μ_∞ is replaced by μ_1.

Corollary 5.4. *Let $S\in\mathcal{V}$. Then* $\operatorname{stab}(\operatorname{Re} S)<0 \iff \mu(\operatorname{Re} S)<0$ *for some Lozinskiĭ measure μ on $\mathcal{V}^n$.*

Theorem 5.5. *For* $\operatorname{stab}(\operatorname{Re} S)<0$ *it is sufficient and necessary that* $\operatorname{stab}(\operatorname{Re} S^{[2]})<0$ *and* $(-1)^n\det(\operatorname{Re} S)>0$.

Proof. By the spectral property of $S^{[2]}$, the condition $\operatorname{stab}(\operatorname{Re} S^{[2]})<0$ implies that at most one eigenvalue of S can have a nonnegative real part. We may thus suppose that all eigenvalues are real. It is then simple to see that the existence of one and only one nonnegative eigenvalue is precluded by the condition $(-1)^n\det(\operatorname{Re} S)>0$. $\square$

Theorem 5.5 and Corollary 5.4 lead to the following result.

Theorem 5.6. *Suppose that* $(-1)^n \det(\operatorname{Re} S) > 0$. *Then* S *is stable if and only if* $\mu(\operatorname{Re} S^{[2]}) < 0$ *for some Lozinskiĭ measure* μ *on* $\mathcal{V}^N$, $N = \binom{n}{2}$.

Theorem 5.7. *If* $\operatorname{stab}(\operatorname{Re} S^{[2]}(\beta)) < 0$ *for* $\beta \in (a,b)$, *then* (a,b) *contains no Hopf bifurcation points of* $S(\beta)$.

Proof. Let $\beta \mapsto S(\beta) \in \mathcal{V}$ be a function that is continuous for $\beta \in (a,b)$. A point $\beta_0 \in (a,b)$ is said to be a *Hopf bifurcation point* for $S(\beta)$ is $S(\beta)$ is stable for $\beta < \beta_0$, and there exists a pair of complex eigenvalues $\operatorname{Re}\lambda(\beta) \pm i \operatorname{Im}\lambda(\beta)$ of $S(\beta)$ such that $\operatorname{Re}\lambda(\beta) > 0$, while the rest of the eigenvalues of $S(\beta)$ have nonzero real parts for $\beta > \beta_0$. From the proof of Theorem 5.5 we see that $\operatorname{stab}(\operatorname{Re} S^{[2]}) \le 0$ precludes the existence of a pair of eigenvalues of S having positive real parts. □

Let S and P be $n \times n$ complex cyclic matrices. A subspace $\Omega \in \mathcal{V}$ is *invariant* under S if $S(\Omega) \subset \Omega$. S is said to be *stable* with respect to an invariant subspace Ω if the restriction of S to Ω, $S\Big|_{\Omega} : \Omega \mapsto \Omega$ is stable. Let the matrix P be such that $\operatorname{rank} P = r$ $(0 < r < n-1)$ and

$$PS = O. \tag{5.4}$$

Then $\operatorname{Ker} P = \{\mathbf{Z} \in \mathcal{V},\ P\mathbf{Z} = 0\}$ satisfies $S(\mathcal{V}) \subset \operatorname{Ker} P$. In particular, $\operatorname{Ker} P$ is an $(n-r)$-dimensional invariant space of S. It is of interest to study the stability of S with respect to $\operatorname{Ker} P$ when (5.4) holds.

Lemma 5.8. *Let* $\Omega \subset \mathcal{V}$ *be a subspace such that* $S(\mathcal{V}) \subset \Omega$ *and* $\dim \Omega = k < n$. *Then* 0 *is an eigenvalue of* S, *and there exist* $n-k$ *null eigenvectors that do not belong to* Ω.

Proof. Let W be the quotient space $\mathcal{V}/\Omega$. Then $\mathcal{V} \cong \Omega \oplus W$ and $S(W) = \{0\}$ since $S(\mathcal{V}) \subset \Omega$. This establishes the lemma. □

Theorem 5.9. *Suppose that* P *and* S *satisfy* (5.4) *and* $\operatorname{rank} P = r$ $(0 < r < n-1)$. *Then for* S *to be stable with respect to* $\operatorname{Ker} P$, *it is necessary and sufficient that*

1°. $\operatorname{stab}(\operatorname{Re} S^{[r+2]}) < 0$, *and*

2°. $\limsup_{\varepsilon \to 0^+} \operatorname{sign}[\det \operatorname{Re}(\varepsilon I + S)] = (-1)^{n-r}$.

Proof. Let λ_i $(1 \le i \le n-r)$ be eigenvalues of $S\Big|_{\operatorname{Ker} P}$. By Lemma 5.8, the eigenvalues of S can be written as

$$\lambda_1, \lambda_2, \ldots, \lambda_{n-r}, \underbrace{0, \ldots, 0}_{r},$$

and thus $\{\lambda_i + \lambda_j,\ 1 \le i < j \le n-r\} \subset \sigma(S^{[r+2]})$ by the spectral property of additive compound matrices discussed in §2. It follows that

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$\operatorname{stab}(\operatorname{Re} S^{[r+2]}) < 0$ precludes the possibility of more than one λ_i $(1 \le i \le n-r)$ having nonnegative real parts. For $\varepsilon > 0$ sufficiently small

$$\operatorname{sign}[\det \operatorname{Re}(\varepsilon I + S)] = \operatorname{sign}(\varepsilon^r \lambda_1 \cdots \lambda_{n-r}).$$

The theorem can be proved using the same arguments as in the proof of Theorem 5.5. □

Remark 5.10. If $r = n$ in (5.4), then P is of full rank and hence $S = O$. If $r = n-1$, then $\operatorname{Ker} P$ is of dimension 1 and thus the eigenvalues of S are λ_1 and 0 of multiplicity $n-1$. From the above proof, we know that Theorem 5.9 still holds in this case, if condition 1° is replaced by $\operatorname{tr}(\operatorname{Re} S) < 0$.

Corollary 5.11. *Suppose that S and P_1 satisfy*

$$P_1 S = \beta P_1 \tag{5.5}$$

and $\operatorname{rank} P_1 = r$ $(0 < r < n-1)$. *Thus S is stable with respect to* $\operatorname{Ker} P_1$ *if and only if the following conditions hold:*

1°. $\operatorname{stab}(\operatorname{Re} S^{[r+2]}) < (r+2)\beta$, *and*
2°. $(\operatorname{sign}\beta)^r(-1)^{n-r}\det(\operatorname{Re} S) > 0$.

Proof. Let the matrix P_1 be such that $\operatorname{rank} P_1 = r$ $(0 < r < n-1)$ and (5.5) holds for some scalar $\beta \ne 0$. Then $\operatorname{Ker} P_1$ is an invariant subspace of S. Noting that (5.5) is equivalent to $P_1(S - \beta I) = O$, one can apply Theorem 5.9 to $S - \beta I$ and obtain the proof. □

6. Remarks

1°. Actually, the quasicyclic complex vector functional equation has broken cyclicity and therefore it is not possible to find its reproductive solution.
2°. The quasicyclic complex vector functional equation solved here as a particular case for $n = 3$ has strong modifying properties. During the solution of the equation we always took into account a cyclic permutation of two vectors in its right-hand side, such that the equation preserves quasicyclicity. In the opposite case, for another kind of cyclic permutation of the vectors, the equation transforms into a semicyclic or noncyclic functional equation which needs to be solved over again. This shows that a quasicyclic functional equation changes its stucture so that it goes over from one to another class, *i.e.*, it appears in different modifications, and accordingly we adapt the method for its solution.
3°. According to the above description, by the same procedure which is proposed in this paper it is possible to solve the following complex vector functional equations

$$\begin{aligned} & a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) + a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1) + a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2) \\ = \; & \alpha_{11} f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_3) + \alpha_{12} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1) + \alpha_{13} f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_2), \end{aligned} \tag{6.1}$$

$$\begin{aligned} &a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ &\alpha_{21} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\alpha_{22} f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_2)+\alpha_{23} f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_3), \end{aligned} \tag{6.2}$$

$$\begin{aligned} &a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ &\alpha_{31} f(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_{32} f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_{33} f(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_3), \end{aligned} \tag{6.3}$$

$$\begin{aligned} &a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ &\alpha_{41} f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_{42} f(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_{43} f(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_3), \end{aligned} \tag{6.4}$$

$$\begin{aligned} &a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ &\alpha_{51} f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_{52} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_{63} f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_3), \end{aligned} \tag{6.5}$$

$$\begin{aligned} &a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ &\alpha_{61} f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_{62} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_{63} f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_3), \end{aligned} \tag{6.6}$$

$$\begin{aligned} &a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)+a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_1)+a_3 f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_2)\\ = \ &\alpha_{11} f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_{12} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_3)+\alpha_{13} f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_1)\\ + \ &\alpha_{21} f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_3)+\alpha_{22} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_1)+\alpha_{23} f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_2)\\ + \ &\alpha_{31} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_1)+\alpha_{32} f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_2)+\alpha_{33} f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_3)\\ + \ &\alpha_{41} f(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_1)+\alpha_{42} f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_2)+\alpha_{43} f(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_3)\\ + \ &\alpha_{51} f(\mathbf{Z}_2,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_{52} f(\mathbf{Z}_3,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_{53} f(\mathbf{Z}_1,\mathbf{Z}_3,\mathbf{Z}_3)\\ + \ &\alpha_{61} f(\mathbf{Z}_3,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_{62} f(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_{63} f(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_3)\\ + \ &\alpha_{71} f(\mathbf{Z}_1,\mathbf{Z}_1,\mathbf{Z}_1)+\alpha_{72} f(\mathbf{Z}_2,\mathbf{Z}_2,\mathbf{Z}_2)+\alpha_{73} f(\mathbf{Z}_3,\mathbf{Z}_3,\mathbf{Z}_3), \end{aligned} \tag{6.7}$$

where $a_i,\ \alpha_{ij}$ are complex constants.
4°. The above functional equations are not more than different modifications of the quasicyclic functional equation considered.

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7. Some open research problems

By virtue of the results obtained here, we naturally come to the idea of formation of new general classes of quasicyclic functional equations whose solution will be of interest. In this direction, we will give the following general quasicyclic complex vector functional equations, with the same notations used in this article and $\mathbf{Z}_{n+i} \equiv \mathbf{Z}_i$ (only in 1–2), with an intention to be considered as open problems:

(1) $$\sum_{i=1}^{k} a_i f(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) = \sum_{i=1}^{k} \alpha_i \mathcal{C}^{i-1} f(\mathbf{Z}_i, \mathbf{Z}_i, \dots, \mathbf{Z}_i, \mathbf{Z}_{i+1}),$$
$$(p < n < 2p-1,\ k \le n),$$

where $\mathcal{C}$ is a cyclic operator such that

$$\mathcal{C} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) = f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_1).$$

(2) $$\sum_{i=1}^{k} a_i f(\mathbf{X}_i, \mathbf{X}_{i+1}, \dots, \mathbf{X}_{i+p-1}, \mathbf{Y}_i, \mathbf{Y}_{i+1}, \dots, \mathbf{Y}_{i+q-1})$$
$$= \sum_{i=1}^{k} \alpha_i \mathcal{C}^{i-1} f(\mathbf{X}_i, \mathbf{X}_i, \dots, \mathbf{X}_i, \mathbf{X}_{i+1}, \mathbf{Y}_i, \mathbf{Y}_i, \dots, \mathbf{Y}_i, \mathbf{Y}_{i+1})$$
$$(k \le n;\ p \le n,\ q \le n),$$

where $\mathcal{C}$ is as above.

(3) $$(-1)^n a_{n+1} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) - a_{n+2} f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_n, \mathbf{Z}_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^{i+1} a_i f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_i + \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{n+1})$$
$$= (-1)^n \alpha_{n+1} f(\mathbf{Z}_{n+1}, \dots, \mathbf{Z}_{n+1}) - \alpha_{n+2} f(\mathbf{Z}_1, \dots, \mathbf{Z}_1)$$
$$+ \sum_{i=1}^{n} (-1)^{i+1} \alpha_i \mathcal{C}^{i-1} f(\mathbf{Z}_i, \mathbf{Z}_i, \dots, \mathbf{Z}_i, \mathbf{Z}_{i+1}),$$

where $\mathcal{C}$ is as above.

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Ice B. Risteski
2 Milepost Place # 606
Toronto, M4H 1C7, Ontario, Canada
e-mail address: ice@scientist.com

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